Mixed recurrence relations and interlacing properties for zeros of q-classical orthogonal polynomials

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Computer Algebra System

We use the computer algebra system Maple

Algorithm

We use a modified version of the q-analogue of Zeilberger's algorithm to get our mixed recurrence equations.

Application

Among the families of classical q-orthogonal polynomials, we will consider the little q-Jacobi family to illustrate our interlacing results.

Consider for example the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k.$$

They are solutions of a three-term recurrence equation

$$(n+1)L_{n+1}^{(\alpha)}(x) + (x - (2n+\alpha+1))L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0.$$

Question: How can we get such kinds of recurrence equations?
A computational solution is "using the Zeilberger algorithm" implemented in the Maple `hsum` package (see [Koepf, 2014]). Maple

Zeilberger Algorithm

Recurrence Equations for Hypergeometric Series

Doron Zeilberger (1990) designed an algorithm to compute recurrence equations for hypergeometric sums of the type

$$s_n = \sum_{k=-\infty}^{\infty} F(n,k).$$

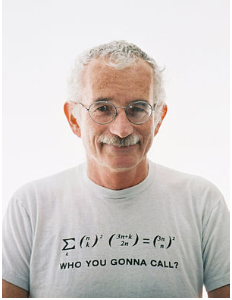
The summand $F(n,k)$ must be a hypergeometric term w. r. t. both variables n and k, that is,

$$\frac{F(n,k+1)}{F(n,k)} \in \mathbb{Q}(n,k), \ \frac{F(n+1,k)}{F(n,k)} \in \mathbb{Q}(n,k).$$

Zeilberger Algorithm

Holonomic Recurrence Equations

Zeilberger's algorithm results in a holonomic recurrence equation for s_n. A recurrence equation is called holonomic if it is linear, homogeneous, and has polynomial coefficients.



Recurrence equation for $s_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{3n+k}{2n}$

Maple

q-version of Zeilberger's algorithm

The q-analogues of Zeilberger's algorithms:

- is implemented in the Maple `qsum` package (see [Koepf, 2014])
- deals with definite sums of the form

$$S_n = \sum_{k=-\infty}^{\infty} A(n,k)$$

- applies if $A(n,k)$ is a q-hypergeometric term with respect to both n and k. That is, $\frac{A(n,k+1)}{A(n,k)} \in \mathbb{Q}(q^n, q^k)$, $\frac{A(n+1,k)}{A(n,k)} \in \mathbb{Q}(q^n, q^k)$
- generates a q-holonomic recurrence equation for S_n: a linear homogeneous recurrence equation

$$\sum_{k=0}^{n} \alpha_k(q;q^m) C_{m+k} = 0$$

is called q-holonomic if the coefficients $\alpha_k(q;q^m)$ are rational w.r.t. q and polynomial functions w.r.t. the variable q^m

The Little q-Jacobi polynomials

For example, the Little q-Jacobi polynomials $p_n(x;\alpha,\beta|q)$ have the series representation

$$\tilde{p}_n(x;\alpha,\beta|q) = (-1)^n q^{\binom{n}{2}} \frac{(\alpha q;q)_n}{(\alpha\beta q q^n;q)_n} \sum_{k=0}^{n} \frac{\left(q^{-n},\alpha\beta q^{n+1};q\right)_k (qx)^k}{\left(\alpha q,q;q\right)_k},$$

with $0<\alpha q<1$, $\beta q<1$, $x\in(0,1)$, where $(a,b;q)_k=(a;q)_k(b;q)_k$ is the q-Pochhammer symbol with

$$(a;q)_0=1,\ (a;q)_k=(1-a)(1-aq)\times\cdots\times(1-aq^{k-1}),\ \text{if } k=1,2,\ldots.$$

Using the q-version of Zeilberger's algorithm, we get the recurrence equation they satisfy: Maple

A mixed recurrence equation

We are interested by mixed recurrence equations like

$$\tilde{p}_n(x;\alpha,\beta|q) = \tilde{p}_n(x;\alpha q,\beta|q) + \frac{\alpha\, q^n\,(q^n-1)\,(\beta\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{p}_{n-1}(x;\alpha q,\beta|q).$$

How can we obtain such kind of equations?

A computational approach answer is that we used an adaption of the `qsumdiffeq` procedure (see [Koepf, 2014]) of the `qsum` package. We propose two programs for this purpose:

The first program

The first program called `qMixRec1(F,q,k,S(n),a,s)` finds a mixed recurrence equation of the form

$$S(n,a)=\sum_{j=0}^{J}\sigma_j S(n-j,aq^s),J=1,2,\ldots,$$

where $S(n,a)=\sum_{k=-\infty}^{\infty}F(n,k,a)$, and s is a positive integer.
For example if we write

$$\tilde{p}_n(x;\alpha,\beta|q)=\sum_{k=0}^{n}F(n,k,\alpha),\ a=\alpha,\ s=1,$$

we get

$$\tilde{p}_n(x;\alpha,\beta|q)=\tilde{p}_n(x;\alpha q,\beta|q)+\frac{\alpha\, q^n\,(q^n-1)\,(\beta\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{p}_{n-1}(x;\alpha q,\beta|q).$$

The second program

The second one denoted by `qMixRec2(F,q,k,S(n),a,`s_0`,b,`s_1`,`s_2`,r)` finds a mixed recurrence equation of the form

$$S(n,a,bq^{s_1}) = \sum_{j=0}^{J} \sigma_j S(n-j, aq^{s_0}, bq^{s_2+rj}),\ J = 1,2,\dots,\ r \in \{0,1\},$$

where $S(n,a,b) = \sum_{k=-\infty}^{\infty} F(n,k,a,b)$, and $s_0,\ s_1,\ s_2$ are positive integers.

For

$$\tilde{p}_n(x;\alpha,\beta|q) = \sum_{k=0}^{n} F(n,k,\alpha,\beta),\ a=\alpha,\ b=\beta,\ s_0=1,\ s_1=1,\ s_2=0,\ r=1,$$

we get

$$\tilde{p}_n(x;\alpha,\beta q|q) = \tilde{p}_n(x;\alpha q,\beta|q) + \frac{\alpha\, q^n\,(q^n-1)}{\beta\,\alpha\, q^{2\,n+1}-1}\tilde{p}_{n-1}(x;\alpha q,\beta q|q).$$ *Maple*

How can we apply such mixed recurrence equations?

Interlacing properties

Theorem 1

Let $0 < \alpha q < 1$ and $\beta q < 1$ and denote:

- the zeros of $\tilde{p}_n(x; \alpha, \beta|q)$ by $0 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$,
- the zeros of $\tilde{p}_n(x; \alpha q, \beta|q)$ by $0 < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < 1$,
- the zeros of $\tilde{p}_n(x; \alpha, \beta q|q)$ by $0 < z_{n,1} < z_{n,2} < \cdots < z_{n,n} < 1$,
- and the zeros of $\tilde{p}_{n-1}(x; \alpha q, \beta q|q)$ by $0 < t_{n-1,1} < t_{n,2} < \cdots < t_{n-1,n-1} < 1$.

Then, for $i \in \{1, 2, \ldots, n-1\}$,

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,

(b) $z_{n,i} < y_{n,i} < t_{n-1,i} < z_{n,i+1} < y_{n,i+1}$ if $\beta < 0$.

Theorem 2

Let (c, d) be a finite or infinite interval and assume that p_n and q_n are monic polynomials (not necessarily orthogonal) of degree n, with zeros $c < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < d$ and $c < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < d$, respectively, satisfying the interlacing property

$$x_{n,1} < y_{n-1,1} < x_{n,2} < y_{n-1,2} < \cdots < x_{n,n-1} < y_{n-1,n-1} < x_{n,n}.$$

Assume that a and b are continuous and have constant sign on (c, d) and that f_n is a polynomial of degree n with zeros $c < z_{n,1} < z_{n,2} < \cdots < z_{n,n} < d$, satisfying

$$f_n(x) = a(x)p_n(x) + b(x)q_{n-1}(x).$$

Then, for each $k \in \{1, 2, \ldots, n-1\}$,

(a) if $a(x)$ and $b(x)$ have the same sign on (c, d), $z_{n,k} < x_{n,k} < y_{n-1,k} < z_{n,k+1} < x_{n,k+1}$;

(b) if $a(x)$ and $b(x)$ differ in sign on (c, d), $x_{n,k} < z_{n,k} < y_{n-1,k} < x_{n,k+1} < z_{n,k+1}$.

A proof of Theorem 1

Since $\tilde{p}_n(x;\alpha q,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha q,\beta|q)$ belong to the same orthogonal sequence, their zeros satisfy the interlacing property

$$x_{n,1} < y_{n-1,1} < x_{n,2} < y_{n-1,2} < \cdots < x_{n,n-1} < y_{n-1,n-1} < x_{n,n}.$$

Equation

$$\tilde{p}_n(x;\alpha,\beta|q) = \tilde{p}_n(x;\alpha q,\beta|q)+\frac{\alpha\, q^n\,(q^n-1)\,(\beta\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{p}_{n-1}(x;\alpha q,\beta|q).$$

is in the form of

$$f_n(x) = a(x)p_n(x) + b(x)q_{n-1}(x)$$

with $a(x) = 1 > 0$ and $b(x) = \frac{\alpha\, q^n\left(q^n-1\right)\left(\beta\, q^n-1\right)}{(\alpha\beta\, q^{2\,n+1}-1)(\alpha\,\beta\, q^{2\,n}-1)}$. Taking into consideration the conditions $0 < q < 1$, $0 < \alpha q < 1$, $\beta q < 1$, $b(x) > 0$. Therefore $a(x)$ and $b(x)$ have the same sign and the result follows from the above Theorem 2.

I would like to thank you very much for your interest!