Eliminationsverfahren für nicht-lineare PDE-Systeme

Daniel Robertz

Centre for Mathematical Sciences

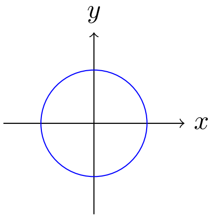
Plymouth University

RESEARCH WITH PLYMOUTH UNIVERSITY

Kassel, 04/05/2017

Algebraische Geometrie

$$\begin{cases} \mathbb{R} \to \mathbb{R}^2 \\ \;t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \end{cases}$$



$$x^2 + y^2 - 1 = 0$$

Eliminiere t in $$x = \frac{2t}{t^2+1}, \quad y = \frac{t^2-1}{t^2+1} \qquad \dots$$

Spezielle Lösungen

$$\begin{aligned} \frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \frac{1}{\rho} \nabla p &= 0 \qquad \text{(Navier-Stokes)} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0 \end{aligned}$$

Zylinderkoordinaten $r,\ \theta,\ z,$ $\rho \equiv 1$ (inkompressibler Fluss)

Ansatz: $v_i(r, \theta, z) = f_i(r) g_i(\theta) h_i(z),$ $i = 1,\ 2,\ 3$

PDE: $u u_{x,y} - u_x u_y = 0,$ $u \in \{v_1, v_2, v_3\},$ $(x, y) \in \{(r, \theta), (r, z), (\theta, z)\}$

eines der vielen einfachen Systeme der Thomas-Zerlegung:

$$\begin{aligned} v(t, r, \theta, z) &= \left(-\frac{(t+c_2) F_1(t)}{r} - \frac{r}{2(t+c_2)},\ \frac{(\theta+c_1) r}{t+c_2},\ 0 \right), \\ p(t, r, \theta, z) &= (t + c_2) \ln(r) \dot{F}_1(t) - \frac{(t+c_2)^2 F_1(t)^2}{2r^2} + (\ln(r) + (\theta + c_1)^2) F_1(t) \\ &\quad + F_2(t) - \frac{2\nu \ln(r)}{t+c_2} + \frac{((\theta+c_1)^2 - \frac{3}{4}) r^2}{2(t+c_2)^2}. \end{aligned}$$

Outline

- Thomas-Zerlegung für Differentialsysteme
- Implizitisierungsproblem
- Eliminationsverfahren

1. Thomas-Zerlegung für Differentialsysteme

Systeme von linearen PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\[2ex] \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{array} \right.$$

bestimme: $u = u(x, y)$ analytisch

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + {\color{red} a_{0,1}}\, y + {\color{red} a_{2,0}}\, \frac{x^2}{2!} + {\color{red} a_{1,1}}\, \frac{xy}{1!1!} + {\color{red} a_{0,2}}\, \frac{y^2}{2!} + \ldots$$

Janet-Algorithmus bestimmt Vektorraum-Basis für Potenzreihenlösungen

(Maurice Janet, $\sim$ 1920)

Janet-Algorithmus

$$\left\{\begin{array}{rcll} u_{y,y} & = & 0 & \qquad A \\[2ex] u_{x,x} - y u_{z,z} & = & 0 & \qquad B \end{array}\right.$$

ist äquivalent zu

$$\left\{\begin{array}{rcll} u_{y,y} & = & 0 & \qquad A \\[2ex] u_{x,x} - y u_{z,z} & = & 0 & \qquad B \\[2ex] u_{y,z,z} & = & 0 & \qquad \frac{1}{2}(\partial_x^2 - y\partial_z^2)A - \frac{1}{2}\partial_y^2 B \\[2ex] u_{x,y,y} & = & 0 & \qquad \partial_x A \\[2ex] u_{z,z,z,z} & = & 0 & \qquad \frac{1}{2}(\partial_x^4 - 2y\partial_x^2\partial_z^2 + y^2\partial_z^4)A - \frac{1}{2}(\partial_x^2\partial_y^2 - y\partial_y^2\partial_z^2 + 2\partial_y\partial_z^2)B \\[2ex] u_{x,y,z,z} & = & 0 & \qquad \frac{1}{2}(\partial_x^3 - y\partial_x\partial_z^2)A - \frac{1}{2}\partial_x\partial_y^2 B \\[2ex] u_{x,z,z,z,z} & = & 0 & \qquad \frac{1}{2}(\partial_x^5 - 2y\partial_x^3\partial_z^2 + y^2\partial_x\partial_z^4)A - \frac{1}{2}(\partial_x^3\partial_y^2 + y\partial_x\partial_y^2\partial_z^2 - 2\partial_x\partial_y\partial_z^2)B \end{array}\right.$$

Taylor-Koeff. zu 1, z, y, x, z^2, yz, xz, xy, z^3, xz^2, xyz, xz^3 *frei wählbar*,
alle anderen dadurch bestimmt (lineare Gleichungen)

Polynomiale ODEs / PDEs

$$\left(\frac{du}{dt}\right)^2 - 4\,t\,\frac{du}{dt} - 4\,u + 8\,t^2 = 0$$

bestimme: $u = u(t)$ analytisch

$$u(t) = a_0 + a_1\,t + a_2\,\tfrac{t^2}{2!} + a_3\,\tfrac{t^3}{3!} + \ldots$$

Einsetzen und Koeffizientenvergleich:

$$\left\{\begin{array}{ll} a_1^2 - 4\,a_0 = 0 & \qquad a_0 := 0 \quad \Rightarrow \quad a_1 = 0 \\ 2\,a_1\,a_2 - 8\,a_1 = 0 & \\ a_1\,a_3 + a_2^2 - 6\,a_2 + 8 = 0 & \qquad \Rightarrow \quad (a_2 - 2)(a_2 - 4) = 0 \\ \vdots & \end{array}\right.$$

Viele Fallunterscheidungen?

Thomas-Algorithmus $\rightsquigarrow$ endlich viele sog. einfache Systeme

(Joseph Miller Thomas, $\sim$ 1930)

Algebraische Geometrie

$$L = \{\ p_1(x_1, ..., x_n) = 0,\ ...,\ p_r = 0,\ q_1 \neq 0,\ ...,\ q_s \neq 0\ \}$$

polynomiale Gleichungen (und Ungleichungen)

$$\mathrm{Sol}(L) = \{\ a \in \mathbb{C}^n \mid p_i(a) = 0,\ q_j(a) \neq 0\ \forall\, i, j\ \}$$

Umgekehrt sei $S \subseteq \mathbb{C}^n$.

$$\mathcal{I}(S) = \{\ p \in \mathbb{C}[x_1, ..., x_n] \mid p(a) = 0\ \forall\, a \in S\ \}$$

Nullstellensatz (Hilbert, 1893) (für Gleichungen)

$$\text{Radikalideale von } \mathbb{C}[x_1, ..., x_n] \quad \underset{\mathcal{I}}{\overset{\mathrm{Sol}}{\rightleftarrows}} \quad \text{Nullstellenmengen in } \mathbb{C}^n$$

sind zueinander inverse Bijektionen.

Differential-algebraische Geometrie

Differentialalgebra (Ritt, Kolchin, Seidenberg, ...)

$\mathbb{Q} \subseteq K$ Differentialkörper mit kommutierenden Derivationen ∂_1, ..., ∂_n

Differentialpolynomring mit Derivationen ∂_1, ..., ∂_n

$K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1,z_1}, ...]$

$K\{u\}$ nicht Noethersch (z.B. $[u'u'', u''u''', \ldots] \subseteq K\{u\}$ nicht endl. erz.)

Satz. (Ritt-Raudenbush).
Jedes Radikal-Differentialideal von $K\{u_1, \ldots, u_m\}$ ist endlich erzeugt, ist Schnitt von endlich vielen Prim-Differentialidealen.

Satz. (Differentieller Nullstellensatz).
Jedes Radikal-Differentialideal $I \subsetneq K\{u_1, \ldots, u_m\}$ hat eine Lösung in einer Diff.-Körpererweiterung von K. Verschwindet $f \in K\{u_1, \ldots, u_m\}$ auf allen Lösungen von I, so ist $f \in I$.

Thomas-Zerlegung für nicht-lineare PDEs

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ Differentialpolynomring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

algebraische Reduktion:

$$p = u_{x,x,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$p \;\to\; r = c \cdot p - u_{x,x,y} \cdot q$$

differentielle Reduktion:

$$p = u_{x,x,y,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$\partial_y\, q = \tfrac{\partial q}{\partial u_{x,x,y}}\, u_{x,x,y,y} + \ldots$$

$$p \;\to\; r = \tfrac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y\, q$$

Reduktion verlangt: Initial $c \neq 0$ und Separante $\frac{\partial q}{\partial u_{x,x,y}} \neq 0$

Thomas-Zerlegung für nicht-lineare PDEs

$R = K\{u_1, \ldots, u_m\}$

Def. *Thomas-Zerlegung* eines Differentialsystems S (oder Sol(S)):

Sol(S) = Sol$(S_1) \uplus \ldots \uplus$ Sol(S_r), S_i einfaches Differentialsystem

Satz. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ einf. Diff.-System

E Differentialideal erzeugt von p_1, ..., p_s

q Produkt der Initiale und Separanten aller p_i

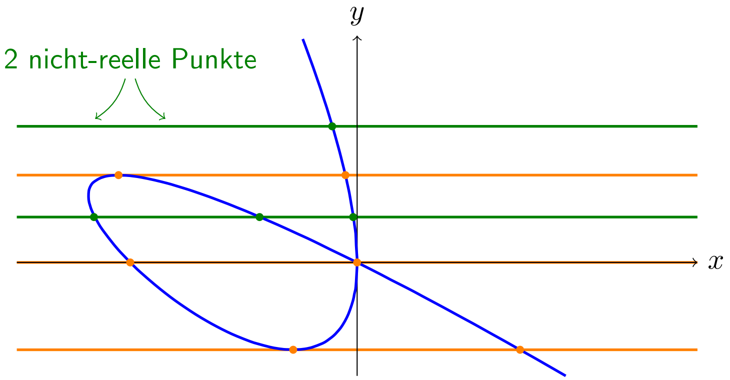
Dann besteht

$$E : q^\infty := \{ p \in R \mid q^r \cdot p \in E \text{ für ein } r \in \mathbb{Z}_{\geq 0} \} = \mathcal{I}_R(\mathrm{Sol}(S))$$

aus allen Differentialpolynomen in R, die auf Sol(S) verschwinden.

Thomas-Zerlegung

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Thomas-Zerlegung

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x,c,b,a], \qquad x > c > b > a$$

$$\boxed{\begin{array}{rcl} a\underline{x}^2 + b\underline{x} + c & = & 0 \\ 4a\underline{c} - b^2 & \neq & 0 \\ & & \\ a & \neq & 0 \end{array}} \quad \boxed{\begin{array}{rcl} 2a\underline{x} + b & = & 0 \\ 4a\underline{c} - b^2 & = & 0 \\ & & \\ a & \neq & 0 \end{array}} \quad \boxed{\begin{array}{rcl} b\underline{x} + c & = & 0 \\ & & \\ b & \neq & 0 \\ a & = & 0 \end{array}} \quad \boxed{\begin{array}{rcl} & & \\ c & = & 0 \\ b & = & 0 \\ a & = & 0 \end{array}}$$

$x_1 \neq x_2$ $\qquad$ $x_1 = x_2$ $\qquad$ x_1 $\qquad$ alle $x \in \overline{\mathbb{Q}}$

lösen $\; p(x) = 0 \;$ für feste Wahl von a, b, c

Thomas-Zerlegung

$$S = \{\, p_1 = 0, \, \ldots, \, p_s = 0, \, q_1 \neq 0, \, \ldots, \, q_t \neq 0 \,\}$$

Def. *Thomas-Zerlegung* eines Differentialsystems S (oder $\mathsf{Sol}(S)$):

$\mathsf{Sol}(S) = \mathsf{Sol}(S_1) \uplus \ldots \uplus \mathsf{Sol}(S_r)$, S_i einfaches Differentialsystem

Def. S ist *einfach*, wenn

(a) $p_1, \, \ldots, \, p_s, \, q_1, \, \ldots, \, q_t$ haben paarweise verschiedene Leitvariablen,

(b) Leitkoeff. und Diskriminanten von p_i und q_j verschwinden nicht,

(c) $p_1, \, \ldots, \, p_s$ bilden passives PDE-System,

(d) $q_1, \, \ldots, \, q_t$ sind reduziert modulo $p_1, \, \ldots, \, p_s$.

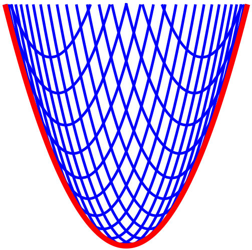
zulässige Derivationen $\mu_i \subseteq \{\partial_1, \ldots, \partial_n\}$ für p_i, $i = 1, \, \ldots, \, s$

Thomas-Zerlegung

$$p = \dot{u}^2 - 4t\dot{u} - 4u + 8t^2 = 0 \qquad\qquad p \in \mathbb{Q}(t)\{u\}$$

Separante von p: $\quad \frac{\partial p}{\partial \dot{u}} = 2\dot{u} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{u}}, \dot{u}) = -16u + 16t^2$



Thomas-Zerlegung:

$$\boxed{\begin{array}{rcl} p & = & 0 \\ u - t^2 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} & & \\ u - t^2 & = & 0 \end{array}}$$

allgemeine Lösung: $\quad u(t) = 2((t+c)^2 + c^2), \qquad c \in \mathbb{R}$

wesentliche singuläre Lösung: $\quad u(t) = t^2$

Spezielle Lösungen

$u_t - 6uu_x + u_{x,x,x} = 0$ (Korteweg-de Vries)

Ansatz: $u(t,x) = f(t)g(x)$

PDE: $uu_{t,x} - u_t u_x = 0$

Thomas-Zerlegung von $\{u_t - 6uu_x + \underline{u_{x,x,x}} = 0,\ u\underline{u_{t,x}} - u_t u_x = 0\}$:

$$\boxed{u = 0} \qquad \boxed{\begin{array}{rcl} u_t - 6uu_x &=& 0 \\ u_{x,x} &=& 0 \\ u &\neq& 0 \end{array}} \qquad \boxed{\begin{array}{rcl} u_t &=& 0 \\ u_{x,x,x} - 6uu_x &=& 0 \\ u_{x,x} &\neq& 0 \\ u &\neq& 0 \end{array}}$$

Lösungen: $u(t,x) = \frac{x+c_1}{-6t+c_2}$ $\qquad x = \pm \int_{u(0)}^{u(x)} \frac{dz}{\sqrt{2z^3 - az - b}}$

Thomas-Zerlegung

- 1937: J. M. Thomas: “Differential Systems”.
- 1998: D. Wang: Implementation für algebraische Systeme
- 2007: V. Gerdt: “On decomposition of algebraic PDE systems into simple subsystems”
- 2009: W. Plesken: “Counting solutions of polynomial systems via iterated fibrations”
- seit 2009: Implementationen in Maple für
  - algebraische Systeme (T. Bächler)
  - PDE-Systeme (M. Lange-Hegermann)

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. R.,
Algorithmic Thomas decomposition of algebraic and differential systems,
J. Symbolic Computation 47(10):1233–1266, 2012.

Bibliographie

J. M. Thomas,
Differential Systems,
AMS Colloquium Publications, vol. XXI, 1937.

V. P. Gerdt,
On decomposition of algebraic PDE systems into simple subsystems,
Acta Appl. Math., 101(1-3):39–51, 2008.

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. Robertz,
Algorithmic Thomas Decomposition of Algebraic and Differential Systems, J. Symbolic Computation 47(10):1233–1266, 2012.

D. Robertz,
Formal Algorithmic Elimination for PDEs,
Lecture Notes in Mathematics, Vol. 2121, Springer, 2014.

Bibliographie

D. Grigoriev,
Complexity of quantifier elimination in the theory of ordinary differential equations, in: LNCS, vol. 378, 1989, pp. 11–25.

F. Boulier, D. Lazard, F. Ollivier, M. Petitot,
Representation for the radical of a finitely generated differential ideal, ISSAC 1995, pp. 158–166.

D. Wang,
Decomposing polynomial systems into simple systems,
J. Symbolic Computation 25(3):295–314, 1998.

E. Hubert,
Notes on triangular sets and triangulation-decomposition algorithms.
in: LNCS, Vol. 2630, 2003, pp. 1–39 and 40–87.

F. Lemaire, M. Moreno Maza, Y. Xie,
The RegularChains library in Maple, SIGSAM Bulletin 39(3):96–97, 2005.

2. Implizitisierungsproblem

Implizitisierungsproblem

$\Omega \subseteq \mathbb{C}^n$ offen und zusammenhängend, $z_1, \ldots, z_n$ Koordinaten

$\mathcal{O} :=$ Ring der analytischen Funktionen auf Ω, $K := \mathrm{Quot}(\mathcal{O})$

∂_i partieller Differentialoperator bzgl. z_i

$p \in \mathcal{O}[F_1, \ldots, F_k]$ multilinear, $\mathcal{O}[F_1, \ldots, F_k] \subset K\{F_1, \ldots, F_k\}$

$\alpha_i : \Omega \to \mathbb{C}^{\nu(i)}$ analytisch mit Jacobi-Matrix vom Rang $\nu(i)$ auf Ω

$S(p; \alpha_1, \ldots, \alpha_k) := \{\, p(f_1 \circ \alpha_1, \ldots, f_k \circ \alpha_k) \mid f_1, \ldots, f_k \text{ analytisch} \,\}$

implizite Beschreibung von $S = S(p; \alpha_1, \ldots, \alpha_k)$?

Satz. Das Verschwindungsideal von S in $K\{U\}$ ist prim.

Implizitisierungsproblem

Genauer gesagt ist die parametrisierte Familie $S(p; \alpha_1, \ldots, \alpha_k)$ von analytischen Funktionen auf Ω definiert durch

$$\{\, u\colon \Omega \longrightarrow \mathbb{C} \;\mid\; u \text{ ist komplex-analytisch; für fast alle } w \in \Omega \text{ gibt es eine offene Umgebung } \Omega' \subseteq \Omega \text{ von } w \text{ und analytische Funktionen } f_i\colon \alpha_i(\Omega') \to \mathbb{C},\ i = 1, \ldots, k,\ u(z) = p(f_1(\alpha_1(z)), \ldots, f_k(\alpha_k(z))) \text{ für alle } z \in \Omega' \,\}.$$

Beispiel. $\Omega = \mathbb{C}$. Ist x von der Form $f(x^2)$?

Beispiel. $\Omega = \mathbb{C}$. Ist $\sin(x)/x$ von der Form $f(x) \cdot \sin(x)$?

Beispiel (linear)

$f_1(x) \cdot (\cos(x+y))^2 + f_2(y) \cdot \cos(2x+y)$, $\quad$ f_1, f_2 analytisch

$$D_1 := \tfrac{\partial}{\partial y} \circ \tfrac{1}{(\cos(x+y))^2} \qquad \longleftrightarrow \qquad f_1(x) \cdot (\cos(x+y))^2$$

$$D_2 := \tfrac{\partial}{\partial x} \circ \tfrac{1}{\cos(2x+y)} \qquad \longleftrightarrow \qquad f_2(y) \cdot \cos(2x+y)$$

$I := \langle\, D_1 \,\rangle \cap \langle\, D_2 \,\rangle$ $\quad$ Linksideal von $\quad$ $K\langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \rangle$

$$I \qquad \longleftrightarrow \qquad f_1(x) \cdot (\cos(x+y))^2 + f_2(y) \cdot \cos(2x+y)$$

$$D_1 = \tfrac{1}{(\cos(x+y))^2}\, \partial_y + \tfrac{2\sin(x+y)}{(\cos(x+y))^3}, \qquad D_2 = \tfrac{1}{\cos(2x+y)}\, \partial_x + \tfrac{2\sin(2x+y)}{(\cos(2x+y))^2}$$

Implizitisierungsproblem

$\alpha_i : \Omega \to \mathbb{C}^{\nu(i)}$ analytisch mit Jacobi-Matrix vom Rang $\nu(i)$ auf Ω

$S(\alpha_1, g_1; \ldots; \alpha_k, g_k) :=$ Familie von analytischen Funktionen
$f_1(\alpha_1(z))g_1(z) + \ldots + f_k(\alpha_k(z))g_k(z)$

Lemma

Sei $b_1, \ldots, b_{n-\nu(i)} \in K^{n\times 1}$ eine Basis des Kerns von $\left(\frac{\partial \alpha_i}{\partial z}\right)$. Für jede analytische Funktion $u\colon \Omega \to \mathbb{C}$ gilt $u \in S(\alpha_i, g_i)$ g.d.w. u annihiliert wird von

$$\left(\sum_{r=1}^{n} (b_j)_n \, \partial_r\right) \circ \frac{1}{g_i}\,, \qquad j = 1, \ldots, n - \nu(i).$$

Satz

$$S(\alpha_1, g_1; \ldots; \alpha_k, g_k) = \mathrm{Sol}_\Omega(\mathcal{I}(\alpha, g)), \quad \text{wobei} \quad \mathcal{I}(\alpha, g) = \bigcap_{i=1}^{k} \mathcal{I}(\alpha_i, g_i).$$

Zerlegung von Funktionsfamilien

J. d'Alembert (1747):

$$u(x,y) = f(x)\,g(y) \quad \text{hinreichend glatt} \quad \Rightarrow \quad \begin{vmatrix} u & u_x \\ u_y & u_{x,y} \end{vmatrix} = 0$$

C. Stéphanos (ICM 1904): (wahrsch. Annahme: Analytizität; ohne Beweis)

$$u(x,y) = \sum_{i=1}^{n} f_i(x)\,g_i(y) \quad \Longleftrightarrow \quad \begin{vmatrix} u & u_x & \dots & u_{x^n} \\ u_y & u_{x,y} & \dots & u_{x^n,y} \\ \vdots & \vdots & \ddots & \vdots \\ u_{y^n} & u_{x,y^n} & \dots & u_{x^n,y^n} \end{vmatrix} = 0 \qquad (*)$$

F. Neuman (1981): Charakterisierung für C^n-Funktionen

Th. M. Rassias (1984): Gegenbeispiel mit weniger Regularität

weitere Arbeiten: Neuman, Rassias, Šimša, Čadek, Gauchman, Rubel

cf. Rassias, Šimša, *Finite sums decompositions in mathematical analysis*, 1995.

$f_1(w)f_2(x) + f_3(y)f_4(z)$

$$u_{w,y} = u_{w,z} = u_{x,y} = u_{x,z} = 0 \qquad (\Sigma_0)$$

$$u_w\, u_x = 0, \qquad u_y\, u_z = 0 \qquad (\Sigma_1)$$

$$u_{y,z} \neq 0, \quad u_w\, u_x = 0, \quad \begin{vmatrix} u_y & u_{y,y} \\ u_{y,z} & u_{y,y,z} \end{vmatrix} = 0, \quad \begin{vmatrix} u_z & u_{y,z} \\ u_{z,z} & u_{y,z,z} \end{vmatrix} = 0 \qquad (\Sigma_2)$$

$$u_{w,x} \neq 0, \quad u_y\, u_z = 0, \quad \begin{vmatrix} u_w & u_{w,w} \\ u_{w,x} & u_{w,w,x} \end{vmatrix} = 0, \quad \begin{vmatrix} u_x & u_{w,x} \\ u_{x,x} & u_{w,x,x} \end{vmatrix} = 0 \qquad (\Sigma_3)$$

$$u_{w,x} \neq 0, \qquad u_{y,z} \neq 0, \qquad \begin{vmatrix} u & u_w & u_y \\ u_x & u_{w,x} & 0 \\ u_z & 0 & u_{y,z} \end{vmatrix} = 0 \qquad (\Sigma_4)$$

(Σ_0, Σ_1) : $\{f_1(w) + f_3(y)\} \cup \{f_1(w) + f_4(z)\} \cup \{f_2(x) + f_3(y)\} \cup \{f_2(x) + f_4(z)\}$

(Σ_0, Σ_2) : $\{f_1(w) + f_3(y)f_4(z) \mid f_3' \neq 0 \neq f_4'\} \cup \{f_2(x) + f_3(y)f_4(z) \mid f_3' \neq 0 \neq f_4'\}$

(Σ_0, Σ_3) : $\{f_1(w)f_2(x) + f_3(y) \mid f_1' \neq 0 \neq f_2'\} \cup \{f_1(w)f_2(x) + f_4(z) \mid f_1' \neq 0 \neq f_2'\}$

(Σ_0, Σ_4) : $\{f_1(w)f_2(x) + f_3(y)f_4(z) \mid f_1' \neq 0,\ f_2' \neq 0,\ f_3' \neq 0,\ f_4' \neq 0\}$

$f_1(x) f_2(y) + f_3(y) f_4(z)$

$$u_{x,z} = 0 \tag{Σ_0}$$

$$u_x\, u_z = 0, \qquad u_{y,z} = u_{x,y} = 0 \tag{Σ_1}$$

$$u_{y,z} \neq 0, \quad u_{x,y} = 0, \quad \begin{vmatrix} u_y & u_{y,y} \\ u_{y,z} & u_{y,y,z} \end{vmatrix} = 0, \quad \begin{vmatrix} u_z & u_{y,z} \\ u_{z,z} & u_{y,z,z} \end{vmatrix} = 0 \tag{Σ_2}$$

$$u_{x,y} \neq 0, \quad u_{y,z} = 0, \quad \begin{vmatrix} u_x & u_{x,x} \\ u_{x,y} & u_{x,x,y} \end{vmatrix} = 0, \quad \begin{vmatrix} u_y & u_{x,y} \\ u_{y,y} & u_{x,y,y} \end{vmatrix} = 0 \tag{Σ_3}$$

$$u_x \neq 0, \qquad u_z \neq 0, \qquad \begin{vmatrix} u & u_x \\ u_y & u_{x,y} \end{vmatrix} = 0, \qquad \begin{vmatrix} u & u_y \\ u_z & u_{y,z} \end{vmatrix} = 0 \tag{Σ_4}$$

$$\left.\begin{array}{l} u_{x,y} \neq 0, \quad u_{y,z} \neq 0, \quad \begin{vmatrix} u_x & u_{x,x} \\ u_{x,y} & u_{x,x,y} \end{vmatrix} = 0, \quad \begin{vmatrix} u_z & u_{y,z} \\ u_{z,z} & u_{y,z,z} \end{vmatrix} = 0 \\ \qquad \begin{vmatrix} u & u_y & u_{y,y} \\ u_x & u_{x,y} & u_{x,y,y} \\ u_z & u_{y,z} & u_{y,y,z} \end{vmatrix} = 0, \quad \begin{vmatrix} u_x & u_{x,y} \\ u_z & u_{y,z} \end{vmatrix} \neq 0 \end{array}\right\} \tag{Σ_5}$$

$f_1(x)f_2(y) + f_3(y)f_4(z)$

$$(\Sigma_0, \Sigma_1): \quad \{f_1(x) + f_3(y)\} \cup \{f_2(y) + f_4(z)\}$$

$$(\Sigma_0, \Sigma_2): \quad \{f_1(x) + f_3(y)f_4(z) \mid f_3' \neq 0 \neq f_4'\}$$

$$(\Sigma_0, \Sigma_3): \quad \{f_1(x)f_2(y) + f_4(z) \mid f_1' \neq 0 \neq f_2'\}$$

$$(\Sigma_0, \Sigma_4): \quad \{f_2(y)\,(f_1(x) + f_4(z)) \mid f_1' \neq 0 \neq f_4'\}$$

$$(\Sigma_0, \Sigma_5): \quad \{f_1(x)f_2(y) + f_3(y)f_4(z) \mid f_1' \neq 0,\ f_2' \neq 0,\ f_3' \neq 0,\ f_4' \neq 0,\ f_2/f_3 \neq \text{const}\}$$

Bemerkung

Darstellungen sind i.a. nicht eindeutig; z.B.

$$xy + yz + y = (x+1)y + yz = xy + y(z+1) = y(x+z+1).$$

Verschwindungsideal

$p \in \mathcal{O}[F_1, \ldots, F_k]$ multilinear, $\qquad S = S(p; \alpha_1, \ldots, \alpha_k)$

$Z_i :=$ von Annihilator von $f_i \circ \alpha_i$ erzeugtes Differentialideal von $K\{F_i\}$

(lineare Differentialpolynome erster Ordnung)

$Z :=$ von Z_1, ..., Z_k erzeugtes Differentialideal von $K\{F_1, \ldots, F_k\}$

(Prim-Differentialideal)

$\Phi : K\{U\} \longrightarrow K\{F_1, \ldots, F_k\}/Z : U \longmapsto p + Z$

$\mathcal{I}_{K\{U\}}(S) = \ker \Phi$

$K\{F_1, \ldots, F_k\}/Z$ Integritätsbereich $\quad \Rightarrow \quad \mathcal{I}_{K\{U\}}(S)$ prim.

$\mathcal{I}_{K\{U\}}(S)$ endlich erzeugt als Radikal-Differentialideal

Symbolisches Lösen

Anwendung:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

pdsolve (Maple):

allgemeine Lösung $F_1(x) + F_2(y)$ mit Bedingungen an F_i

implizite Beschreibung von $F_1(x) + F_2(y)$: $\frac{\partial^2 U}{\partial x\,\partial y} = 0$

$u(x,y) = (x+y+1)e^{-y}$ ist eine Lösung von (1),

ist jedoch *nicht* von der Form $F_1(x) + F_2(y)$!

Symbolisches Lösen

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad (2)$$

pdsolve (Maple): allg. Lösung $F_1(x) + F_2(y)$, Bedingungen an F_i

Thomas-Zerlegung:

$$\boxed{\begin{array}{rcl} \underline{u_{x,x}}^2 + u_{y,y} + u_x + u_y & = & 0 \\ \underline{u_{y,y}} + u_x + u_y & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} u_{x,x} & = & 0 \\ u_{y,y} + u_x + u_y & = & 0 \end{array}}$$

$$u(x,y) = (c_4\,(x+y+1) - c_3)\,e^{-y} + c_2\,(x-y) + c_1, \qquad c_1,\,\ldots,\,c_4 \in \mathbb{R}$$

z.B. ist $u(x,y) = (x+y+1)e^{-y}$ eine Lösung von (2),

ist jedoch nicht von der Form $F_1(x) + F_2(y)$ (wende $\dfrac{\partial^2}{\partial x\,\partial y}$ an)

Bibliographie

W. Plesken, D. Robertz,
Linear Differential Elimination for Analytic Functions,
Mathematics in Computer Science, 4(2–3):231–242, 2010.

D. Robertz,
Formal Algorithmic Elimination for PDEs,
Lecture Notes in Mathematics, Vol. 2121, Springer, 2014.

Th. M. Rassias, J. Šimša,
Finite sums decompositions in mathematical analysis,
Pure and Applied Mathematics, John Wiley & Sons, 1995.

P. J. Olver and P. Rosenau,
The construction of special solutions to partial differential equations,
Physics Letters 114A:3 (1986), 107–112.

Bibliographie

Xiao-Shan Gao,
Implicitization of differential rational parametric equations,
J. Symbolic Computation 36(5):811–824, 2003.

G. Carra-Ferro,
A resultant theory for ordinary algebraic differential equations,
in: LNCS, vol. 1255, 1997, pp. 55-65.

S. L. Rueda, J. R. Sendra,
Linear complete differential resultants and the implicitization of linear DPPEs,
J. Symbolic Computation 45(3):324–341, 2010.

S. L. Rueda,
A perturbed differential resultant based implicitization algorithm for linear DPPEs,
J. Symbolic Computation 46:977–996, 2011.

3. Eliminationsverfahren

Implizitisierung

(a) $I :=$ Diff.-Ideal von $(K\{F_1, \ldots, F_k\}/Z)\{U\}$ erz. von $U - (p + Z)$,
berechne $\mathcal{I}_{K\{U\}}(S) = I \cap K\{U\}$ (Diff.-Alg.)

(b) drücke Ableitungen von $u(z) = p(f_1(\alpha_1(z)), \ldots, f_k(\alpha_k(z)))$ bis Ordnung ω in U_J, $F_{j,J}$ aus (für $\partial^J u$, $(\partial^J f_j) \circ \alpha_j$),
berechne $\mathcal{I}_{K\{U\}}(S)_{\leq \omega}$ durch Elimination von $F_{j,J}$ (komm. Alg.)

(c) α_i homogen; $K\{U\}$ mit Standard-Graduierung: $\deg U_J = 1$
Diff.-Monome in U vom Grad d und Ordnung ω erzeugen einen endlich-dim. $\mathbb{C}$-Vektorraum $\mathbb{C}\{U\}_{d,\omega}$
berechne $\mathcal{I}_{K\{U\}}(S) \cap \mathbb{C}\{U\}_{d,\omega}$ durch Koeff.-Vergleich (lin. Alg.)

Differential-Elimination

$R = K\{u_1, \ldots, u_m\}, \quad B_1 \uplus \ldots \uplus B_k = U := \{u_1, \ldots, u_m\}$ Partition

Block ranking: $\quad u_{i_1} \in B_{j_1}, \quad u_{i_2} \in B_{j_2}, \quad J_1, J_2 \in (\mathbb{Z}_{\geq 0})^n$

$$\partial^{J_1} u_{i_1} > \partial^{J_2} u_{i_2} \quad \Longleftrightarrow \quad \begin{cases} j_1 < j_2 \text{ oder } (j_1 = j_2 \text{ und } (\partial^{J_1} > \partial^{J_2} \\ \text{ oder } (J_1 = J_2 \text{ und } \pi^{-1}(u_{i_1}) < \pi^{-1}(u_{i_2}))) \end{cases}$$

Satz. $\quad S$ einfaches Differentialsystem, $\quad 1 \leq i \leq k$

$E_i \subset K\{B_i, \ldots, B_k\}$ von $\{p_1, \ldots, p_s\} \cap K\{B_i, \ldots, B_k\}$ erz. Diff.-Ideal

q_i Produkt der Initiale und Separanten aller p_i im Schnitt

Dann $\qquad (E : q^\infty) \cap K\{B_i, \ldots, B_k\} \; = \; E_i : q_i^\infty.$

Kor. $\qquad \sqrt{E : q^\infty} \cap K\{B_i, \ldots, B_k\} \; = \; (E_1 : q_1^\infty) \cap \ldots \cap (E_r : q_r^\infty).$

Differential-Elimination

$u(w, x, y, z) = f_1(w) f_2(x) + f_3(y) f_4(z)$

$R = K\{f_1, f_2, f_3, f_4, u\}$ mit Derivationen ∂_w, ∂_x, ∂_y, ∂_z

block ranking $>$ mit $\{f_1, f_2, f_3, f_4\} \gg \{u\}$

$$\begin{cases} u - f_1 f_2 - f_3 f_4 = 0, \\ (f_1)_x = (f_1)_y = (f_1)_z = 0, \quad (f_2)_w = (f_2)_y = (f_2)_z = 0, \\ (f_3)_w = (f_3)_x = (f_3)_z = 0, \quad (f_4)_w = (f_4)_x = (f_4)_y = 0. \end{cases}$$

Maple $\rightsquigarrow$ Thomas-Zerlegung bestehend aus 16 einfachen Systemen

Schnitte mit $K\{u\}$ $\rightsquigarrow$ 10 einfache Systeme

Differential-Elimination

$$\begin{cases} u_w \underline{u_{w,w,x}} - u_{w,w} u_{w,x} = 0, \quad u_x \underline{u_{w,x,x}} - u_{w,x} u_{x,x} = 0, \\ u_{w,w,y} = u_{w,w,z} = u_{w,x,y} = u_{w,x,z} = 0, \\ (u u_{w,x} - u_w u_x) \underline{u_{y,z}} - u_y u_z u_{w,x} = 0, \quad u_{w,y} = u_{w,z} = u_{x,y} = u_{x,z} = 0, \\ u \neq 0, \quad u_w \neq 0, \quad u_x \neq 0, \quad u_{w,x} \neq 0, \quad u \underline{u_{w,x}} - u_w u_x \neq 0, \end{cases}$$

$\{\, u(w,x,y,z) = f_1(w) \cdot f_2(x) + f_3(y) \cdot f_4(z) \mid f_1' \neq 0,\ f_2' \neq 0,$
u ist nicht lokal von der Form $F_1(w) \cdot F_2(x)$ mit F_1, F_2 analytisch $\}$.

$$\begin{cases} u_y \underline{u_{y,y,z}} - u_{y,y} u_{y,z} = 0, \quad u_z \underline{u_{y,z,z}} - u_{y,z} u_{z,z} = 0, \\ u_{w,x} = u_{w,y} = u_{w,z} = 0, \quad u_x = 0, \\ u \neq 0, \quad u_y \neq 0, \quad u_z \neq 0, \quad u_{y,z} \neq 0, \quad u \underline{u_{y,z}} - u_y u_z \neq 0, \end{cases}$$

$\{\, u(w,x,y,z) = f_1(w) + f_3(y) \cdot f_4(z) \mid f_3' \neq 0,\ f_4' \neq 0,$
u ist nicht lokal von der Form $F_3(y) \cdot F_4(z)$ mit F_3, F_4 analytisch $\}$.

Differential-Elimination

$$\begin{cases} u_y \underline{u_{y,y,z}} - u_{y,y} u_{y,z} = 0, \quad u_z \underline{u_{y,z,z}} - u_{y,z} u_{z,z} = 0, \\ u_{x,y} = u_{x,z} = 0, \quad u_w = 0, \\ u \neq 0, \quad u_x \neq 0, \quad u_y \neq 0, \quad u_z \neq 0, \quad u_{y,z} \neq 0, \quad u \underline{u_{y,z}} - u_y u_z \neq 0, \end{cases}$$

$$\{\, f_2(x) + f_3(y) \cdot f_4(z) \mid f_2' \neq 0,\ f_3' \neq 0,\ f_4' \neq 0 \,\}.$$

$$u \underline{u_{w,x}} - u_w u_x = 0, \quad u_{w,y} = u_{w,z} = 0, \quad u_y = u_z = 0, \quad u \neq 0,$$

$$\{\, f_1(w) \cdot f_2(x) \mid f_1 \neq 0,\ f_2 \neq 0 \,\}.$$

$$u \underline{u_{y,z}} - u_y u_z = 0, \quad u_w = u_x = 0, \quad u \neq 0,$$

$$\{\, f_3(y) \cdot f_4(z) \mid f_3 \neq 0,\ f_4 \neq 0 \,\}.$$

$$u_{w,x} = u_{w,y} = u_{w,z} = 0, \quad u_x = u_y = 0, \quad u \neq 0, \quad u_z \neq 0,$$

$$\{\, f_1(w) + f_4(z) \mid f_4' \neq 0 \,\}.$$

Differential-Elimination

$$u_{w,x} = u_{w,y} = u_{w,z} = 0, \quad u_x = u_z = 0, \quad u \neq 0,$$

$$\{\, u(w,x,y,z) = f_1(w) + f_3(y) \mid u \neq 0 \,\}.$$

$$u_{x,y} = u_{x,z} = 0, \quad u_w = u_z = 0, \quad u \neq 0, \quad u_x \neq 0,$$

$$\{\, f_2(x) + f_3(y) \mid f_2' \neq 0 \,\}.$$

$$u_{x,y} = u_{x,z} = 0, \quad u_w = u_y = 0, \quad u \neq 0, \quad u_x \neq 0, \quad u_z \neq 0,$$

$$\{\, f_2(x) + f_4(z) \mid f_2' \neq 0,\ f_4' \neq 0 \,\}.$$

$$u = 0.$$

Bibliographie

W. Plesken, D. Robertz,
Elimination for coefficients of special characteristic polynomials,
Experimental Mathematics, 17(4):499–510, 2008.

D. Robertz,
Formal Algorithmic Elimination for PDEs,
Lecture Notes in Mathematics, Vol. 2121, Springer, 2014.

A. Seidenberg,
An elimination theory for differential algebra,
Univ. California Publ. Math. (N.S.), 3:31–65, 1956.

D. Wang,
Elimination practice. Software tools and applications,
Imperial College Press, London, 2004.

Weitere Bibliographie

C. Méray,
Démonstration générale de l'existence des intégrales des équations aux dérivées partielles,
J. de mathématiques pures et appliquées, 3e série, tome VI, 1880.

C. Riquier,
Les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, 1910.

M. Janet,
Leçons sur les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, 1929.

J. M. Thomas,
Differential Systems,
AMS Colloquium Publications, vol. XXI, 1937.

Weitere Bibliographie

J. F. Ritt,
Differential Algebra,
Dover, 1966.

E. R. Kolchin,
Differential algebra and algebraic groups,
Academic Press, 1973.

E. R. Kolchin,
Selected works of Ellis Kolchin with commentary,
AMS, Providence, RI, 1999.
Commentaries by A. Borel, M. F. Singer, B. Poizat, A. Buium,
Ph. J. Cassidy, ed. and with a preface by H. Bass, Buium, Cassidy.

A. Seidenberg,
An elimination theory for differential algebra,
Univ. California Publ. Math. (N.S.), 3:31–65, 1956.

Weitere Bibliographie

J. d'Alembert,
Recherches sur la courbe que forme une corde tenduë mise en vibration,
Histoire de l'académie royale des sciences et belles lettres de Berlin, 3 (1747), 214–249.

Th. M. Rassias,
A criterion for a function to be represented as a sum of products of factors, Bull. Inst. Math. Acad. Sinica, 14(4):377–382, 1986.

F. Neuman, Th. M. Rassias,
Functions decomposable into finite sums of products (old and new results, problems and trends),
In *Constantin Carathéodory: an international tribute, Vol. I, II*,
pp. 956–963. World Sci. Publ., 1991.

Th. M. Rassias, J. Šimša,
Finite sums decompositions in mathematical analysis,
Pure and Applied Mathematics (New York). John Wiley & Sons, 1995.

Weitere Bibliographie

F. Neuman,
Factorizations of matrices and functions of two variables,
Czechoslovak Math. J., 32(107)(4):582–588, 1982.

F. Neuman,
Finite sums of products of functions in single variables,
Linear Algebra Appl., 134:153–164, 1990.

M. Čadek, J. Šimša,
Decomposable functions of several variables,
Aequationes Math., 40(1):8–25, 1990.

H. Gauchman, L. A. Rubel,
Sums of products of functions of x times functions of y,
Linear Algebra Appl., 125:19–63, 1989.

Weitere Bibliographie

W. Plesken, D. Robertz,
Linear Differential Elimination for Analytic Functions,
Mathematics in Computer Science, 4(2–3):231–242, 2010.

W. Plesken, D. Robertz,
Elimination for coefficients of special characteristic polynomials,
Experimental Mathematics, 17(4):499–510, 2008.

W. Plesken, D. Robertz,
Janet's approach to presentations and resolutions for polynomials and linear pdes,
Archiv der Mathematik, 84(1):22–37, 2005.

Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, D. Robertz,
The Maple Package "Janet": I. Polynomial Systems and
II. Linear Partial Differential Equations,
Computer Algebra in Scientific Computing 2003.

Weitere Bibliographie

D. Robertz,
Formal Algorithmic Elimination for PDEs,
Lecture Notes in Mathematics, Vol. 2121, Springer, 2014.

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. Robertz,
Algorithmic Thomas Decomposition of Algebraic and Differential Systems, J. Symbolic Computation 47(10):1233–1266, 2012.

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. Robertz,
Thomas Decomposition of Algebraic and Differential Systems,
In V. P. Gerdt, W. Koepf, E. W. Mayr, and E. H. Vorozhtsov (eds.),
Computer Algebra in Scientific Computing 2010, Tsakhkadzor, Armenia,
LNCS, Vol. 6244, 2010, pp. 31–54.

V. P. Gerdt, D. Robertz,
Lagrangian Constraints and Differential Thomas Decomposition,
Advances in Applied Mathematics 72:113–138, 2016.

Weitere Bibliographie

V. P. Gerdt,
On decomposition of algebraic PDE systems into simple subsystems,
Acta Appl. Math., 101(1-3):39–51, 2008.

W. Plesken,
Counting solutions of polynomial systems via iterated fibrations,
Arch. Math. 92(1):44–56, 2009.

W. Plesken,
Gauss-Bruhat decomposition as an example of Thomas decomposition,
Arch. Math. 92(2):111–118, 2009.

W. Plesken, T. Bächler,
Counting Polynomials for Linear Codes, Hyperplane Arrangements, and Matroids,
Doc. Math. 19:285–312, 2014.

Weitere Bibliographie

Xiao-Shan Gao,
Implicitization of differential rational parametric equations,
J. Symbolic Computation 36(5):811–824, 2003.

G. Carra-Ferro,
A resultant theory for ordinary algebraic differential equations,
in: Lecture Notes Computer Science, vol. 1255, 1997, pp. 55-65.

S. L. Rueda, J. R. Sendra,
Linear complete differential resultants and the implicitization of linear DPPEs,
J. Symbolic Computation 45(3):324–341, 2010.

S. L. Rueda,
A perturbed differential resultant based implicitization algorithm for linear DPPEs,
J. Symbolic Computation 46:977–996, 2011.

Weitere Bibliographie

D. Grigoriev,
Complexity of quantifier elimination in the theory of ordinary differential equations,
in: Lecture Notes Computer Science, 1989, vol. 378, pp. 11–25.

D. Grigoriev,
Complexity of solution of linear systems in rings of differential operators,
J. Mathematical Sciences, 70(4):1873–1880, 1994.

D. Grigoriev,
Newton-Puiseux series for non-holonomic D-modules and factoring linear partial differential operators,
Moscow Math. J. 9:775–800, 2009.

G. J. Reid, A. D. Wittkopf, A. Boulton,
Reduction of systems of nonlinear partial differential equations to simplified involutive forms,
European J. Appl. Math. 7(6):635–666, 1996.

Weitere Bibliographie

E. S. Cheb-Terrab, K. von Bülow,
A computational approach for the analytic solving of partial differential equations, Computer Physics Communications 90 (1995), 102–116.

T. Wolf, A. Brand, M. Mohammadzadeh,
Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions,
J. Symbolic Computation 27(2):221–238, 1999.

T. Wolf,
Applications of CRACK in the classification of integrable systems.
In *Superintegrability in classical and quantum systems*,
CRM Proc. Lecture Notes, Vol. 37, Amer. Math. Soc., 2004, 283–300.

P. J. Olver, P. Rosenau,
The construction of special solutions to partial differential equations,
Physics Letters 114A:3 (1986), 107–112.

Weitere Bibliographie

J.-F. Pommaret,
Partial Differential Equations and Group Theory,
Kluwer Academic Publishers Group, Dordrecht, 1994.

A. Levin,
Dimension polynomials of intermediate fields and Krull-type dimension of finitely generated differential field extensions,
Mathematics in Computer Science 4(2–3):143–150, 2010.

O. Golubitsky, M. Kondratieva, A. Ovchinnikov, A. Szanto,
A bound for orders in differential Nullstellensatz,
J. Algebra 322(11):3852–3877, 2009.

P. Aubry, D. Lazard, M. Moreno Maza,
On the theories of triangular sets,
J. Symbolic Computation 28(1-2):105–124, 1999.

Weitere Bibliographie

L. C. Evans,
Partial differential equations,
Graduate Studies in Mathematics 19, Amer. Math. Soc., 1998.

R. K. Bullough and P. J. Caudrey,
The soliton and its history,
in: R. K. Bullough and P. J. Caudrey (eds.), *Solitons*, volume 17 of *Topics in Current Physics*, Springer, 1980, pp. 1–64.

P. G. Drazin and R. S. Johnson,
Solitons: an introduction,
Cambridge Univ. Press, 1989.

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura,
Method for solving the Korteweg-de Vries equation,
Physical Review Letters 19:19 (1967), 1095–1097.

Weitere Bibliographie

Wu W.-T.,
Mathematics mechanization,
Mechanical geometry theorem-proving, mechanical geometry problem-solving and polynomial equtaions-solving,
Kluwer, 2000.

D. Wang,
Decomposing polynomial systems into simple systems,
J. Symbolic Computation 25(3):295–314, 1998.

Z. Li, D. Wang,
Coherent, regular and simple systems in zero decompositions of partial differential systems,
Syst. Sci. Math. Sci. 12(Suppl.):43–60, 1999.

D. Wang,
Elimination practice. Software tools and applications,
Imperial College Press, London, 2004.

Weitere Bibliographie

F. Boulier, D. Lazard, F. Ollivier, M. Petitot,
Representation for the radical of a finitely generated differential ideal,
ISSAC 1995, pp. 158–166.

E. Hubert,
Notes on triangular sets and triangulation-decomposition algorithms. I. Polynomial systems. II. Differential systems,
in: F. Winkler, U. Langer (eds.), LNCS, Vol. 2630, Springer, 2003.

F. Lemaire, M. Moreno Maza, Y. Xie,
The RegularChains library in Maple,
SIGSAM Bulletin 39(3):96–97, 2005.

C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, W. Pan,
Comprehensive triangular decomposition,
in: Computer Algebra in Scientific Computing, 2007, LNCS, Vol. 4770, Springer.