Computing period matrices of algebraic curves

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Computeralgebra-Tagung Kassel
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Algebraic curves & Riemann surfaces

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In fact, there is a triple category equivalence between

- compact Riemann surfaces
- smooth irreducible projective curves over $\mathbb{C}$
- algebraic function fields in one variable over $\mathbb{C}$

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After these choices we view X as an m-sheeted Riemann surface and $\mathbb{C}(X)$ as an algebraic extension of $\mathbb{C}(x)$ of degree m.

Holomorphic differentials

The space of holomorphic differentials of X is a g-dimensional $\mathbb{C}$-vector space that can be described as

$$\Omega^1(X) = \{h\mathrm{d}x : h \in \mathbb{C}(X) \mid \mathrm{ord}_P(h\mathrm{d}x) \geq 0 \text{ for all } P \in X\}.$$

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$$\frac{1}{y}\mathrm{d}x, \frac{x}{y}\mathrm{d}x, \ldots, \frac{x^{g-1}}{y}\mathrm{d}x \quad \text{if } F \text{ is a hyperelliptic curve}$$

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$$\text{or} \quad \frac{1}{\partial_y f}\mathrm{d}x, \frac{x}{\partial_y f}\mathrm{d}x, \frac{y}{\partial_y f}\mathrm{d}x \quad \text{if } F \text{ is a smooth plane quartic.}$$

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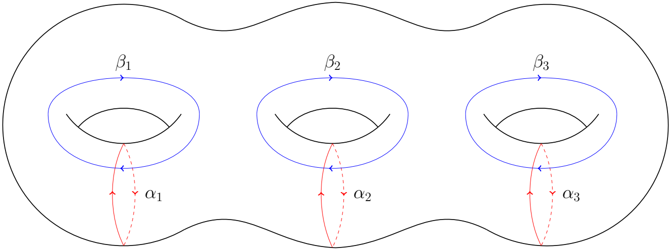

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We obtain a *small period matrix* in the Siegel upper half-space via

$$\tau = \Omega_A^{-1} \Omega_B \in \mathfrak{H}_g,$$

i.e. τ is a symmetric matrix in $\mathbb{C}^{g \times g}$ with positive definite imaginary part.

Jacobian & Abel-Jacobi map

The Jacobian $J = \mathrm{Jac}(X)$ can now be explicitly described as

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By linearity $\mathcal{A}$ can be extended to divisors on X.

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For all of these applications numerical integration has to be performed

- rigorously (provable error bounds)
- to high numerical precision (hundreds or thousands of digits)

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For general algebraic curves there are

- a Maple implementation due to Deconinck and van Hoeij,
- a Python/Sage implementation due to Swierczewski,
- a Matlab implementation due to Frauendiener and Klein,
- a Sage implementation due to Bruin is in progress.

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Compared to the Maple implementation, we compute period matrices

- much faster and more reliably,
- to higher precision,
- for higher genera.

New algorithm for superelliptic curves

Consider a superelliptic curve X given by an affine equation of the form

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- excellent scaling with the genus ($g \gg 1000$ possible)
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- better than Magma for hyperelliptic curves

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- we can integrate along the edges of a spanning tree

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- need to compute only one (!) analytic continuation $y(x) = \sqrt[m]{p(x)}$, which can be done symbolically
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- we can explicitly write down a generating set for $H_1(X, \mathbb{Z})$ and the corresponding intersection matrix
- desingularize integrals between branch points symbolically
  $\to$ integration between branch points possible
- we can integrate along the edges of a spanning tree
  $\to$ only $g \times (n-1)$ integrations

Complexity

Let $X : y^m = p(x)$ be a superelliptic curve of genus g, where $p \in \mathbb{C}[x]$ is separable of degree n.

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The complexity analysis for the computation of the standard period matrices is dominated by linear algebra.

Even the absolute timings are dominated by linear algebra for very large genera.

Timings

Computation* of the big period matrix $\Omega = (\Omega_A, \Omega_B)$ for the family of curves given by

- $(x+y)^{n-1} + x^n y^2 + 1 = 0$ up to 20 significant digits

n	2	3	4	5	6	7	8	9	10
g	1	2	6	10	14	21	28	35	45
Maple	1.7s	5.6s	39s	2m 10s	error	6m 45s	12m 58s	1h 14m	error
(A1)	0.3s	0.9s	3.2s	10s	24s	1m	2m 4s	4m 43s	11m 18s

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- $y^m = x^n + 1$ up to 500 significant digits

(m,n)	(2,5)	(2,11)	(2,31)	(2,101)	(3,5)	(3,11)	(7,11)	(77,11)	(11,31)	(101,31)
g	2	5	15	50	4	10	30	375	150	1500
Maple	17m	1h 4m	-	-	42m	5h	-	-	-	-
(A1)	28s	1m 30s	10m	2h	50s	2m 47s	12m 18s	-	4h 32m	-
Magma	1.6s	6.7s	1m 18s	1h 51m	\	\	\	\	\	\
(A2)	0.15s	0.55s	3.7s	39s	3.8s	11s	14s	1m 35s	1m 32s	44m 32s
(A3)	0.035s	0.14s	1.2s	13s	1.7s	3.4s	3.8s	2m 19s	34s	2h 25m

*done on Intel Xeon(R) CPU E3-1275 V2 3.50GHz processor.

Computations for the LMFDB

During a meeting aimed at expanding the 'L-functions and modular forms database' (LMFDB) to genus 3 curves, (A2) was used, in conjunction with methods of Jeroen Sijsling (et al.), to successfully compute the endomorphism rings of Jacobians of $67,879$ hyperelliptic curves of genus 3.

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For these applications big period matrices were computed to 300 digits precision.

Links

- Our package for superelliptic curves is available at github:

  github.com/pascalmolin/hcperiods

- LMFDB: www.lmfdb.org
- Arb: www.arblib.org