Upper bounds for partial spreads

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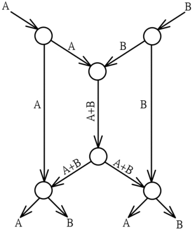


Table for $A_2(11,d;k)$

d\k	2	3	4	5
4	681	97526 - 99718	2383041 - 3370453	18728043 - 27943597
6		290	16669 - 19787	262996 - 328708
8			129 - 132	4097 - 4292
10				65

Computeralgebra Tagung, Kassel, Deutschland, 04.-06.05.2017

Partial spreads

Definition

A *partial* $(k-1)$-*spread* in $\mathrm{PG}(n-1,q)$ is a collection of $(k-1)$-dimensional subspaces with trivial intersection such that each *point* is covered exactly once.

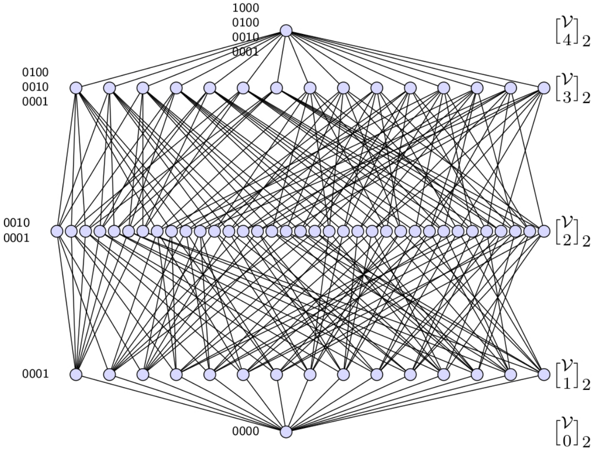
Problem

Determine the maximum size $A_q(n,2k;k)$ of a partial $(k-1)$-spread in $\mathrm{PG}(n-1,q)$.

Remark

A *partial* $(k-1)$-*spread* in $\mathrm{PG}(n-1,q)$ corresponds to a constant dimension code with codewords of dimension k in $\mathbb{F}_q^n$ and subspace distance $2k$.

Subspace lattice of $\mathcal{V} = \mathbb{F}_2^4$



Upper bounds

Drake, Freeman 1979 (Cor. from Bose, Bush 1952)

If $n = kt + r$ with $0 < r < k$, then

$$A_q(n, 2k; k) \leq \sum_{i=0}^{t-1} q^{ik+r} - \lfloor\theta\rfloor - 1 = q^r \cdot \frac{q^{kt} - 1}{q^k - 1} - \lfloor\theta\rfloor - 1,$$

where $2\theta = \sqrt{1 + 4q^k(q^k - q^r)} - (2q^k - 2q^r + 1)$.

Observation

For $r \geq 1$ and $k \geq 2r$ we have $\lfloor\theta\rfloor = \left\lfloor \frac{q^r-2}{2} \right\rfloor$.

If $r = 0$ then $A_q(n, 2k; k) \leq q^r \cdot \frac{q^{kt}-1}{q^k-1}$. (counting points)

If $n < 2k$ then $A_q(n, 2k; k) \leq 1$.

Lower bounds / constructions

$r = 0$: Segre 1964

$A_q(tk, 2k; k) = \frac{q^{tk}-1}{q^k-1}$ for all $t \geq 1$, $k \geq 1$ (matches upper bound – spreads)

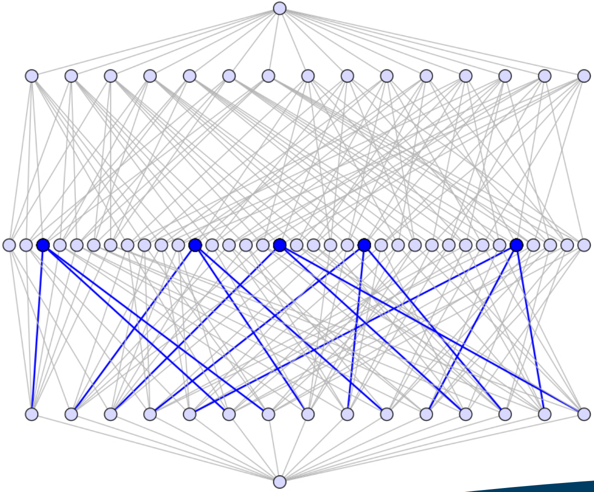
$r \geq 1$: Beutelspacher 1975

$A_q(tk + r, 2k; k) \geq 1 + \sum_{i=1}^{t-1} q^{v-ik} = q^r \cdot \frac{q^{kt}-1}{q^k-1} - q^r + 1$ for all $t \geq 2$, $k \geq 2$ (matches upper bound for $r = 1$)

$r = 2$: El-Zanati, Jordon, Seeliger, Sissokho 2010

$A_2(3m + 2, 6; 3) = \frac{2^{3m+2}-18}{7}$ for all $m \geq 2$ (matches upper bound)

A 1-spread in $PG(3, 2)$



Beutelspacher 1975: study the holes!

- point not covered by a partial spread: hole

Lemma

Let $\mathcal{N}$ be the set of holes of a partial $(k-1)$-spread in $\mathcal{V}$. For every hyperplane $H \subset \mathcal{V}$ we have

$$\#\mathcal{N} \cap H \equiv \#\mathcal{N} \pmod{q^{k-1}}.$$

- point also valid for vector space partitions $[t^{m_t} \ldots k^{m_k} 1^{m_1}]$

Holes and linear codes

- take points of $\mathcal{N}$ as columns of a $v \times n$ matrix G, where $v = \dim(\mathcal{V}$ and $n = \#\mathcal{N}$
- G is generator matrix of a $[n, v]_q$ code $\mathcal{C}$
- the codewords c of $\mathcal{C}$ are:

  $$c = H^T G$$

  for all hyperplanes $H \subset \mathcal{V}$
- $c_i = 0$: point $G_i \in H$; $c_i \neq 0$: point $G_i \notin H$
- the number of non-zeroes in c is equivalent to 0 modulo q^{k-1}
- $\mathcal{C}$ is a q^{k-1}-divisible code
- $\mathcal{N} \cap H$ corresponds to a q^{k-2}-divisible code; recursive

MacWilliams Identities

$$\sum_{j=0}^{n-i} \binom{n-j}{i} A_j = q^{\dim(\mathcal{C})-i} \cdot \sum_{j=0}^{i} \binom{n-j}{n-i} A_j^{\perp} \quad \text{for } 0 \leq i \leq n$$

- A_i: $\#$ codewords of weight i of $\mathcal{C}$
- $A_i^{\perp}$: # codewords of weight i of the dual code $\mathcal{C}^{\perp}$

In our application we have

- $A_0 = A_0^{\perp} = 1$
- $\mathcal{C}$ is projective: $A_1^{\perp} = 0$, $A_2^{\perp} = 0$
- $\mathcal{C}$ is q^{k-1}-divisible: $A_i = 0$ if i is not divisible by q^{k-1}

Linear programming method

If the equation system has no solutions for $A_i, A_i^\perp \in \mathbb{R}_{\geq 0}$, then no such code exists.

Example

There is no 2^3-divisible linear code of length $n = 52$ in $\mathbb{F}_2^v$.

$$\begin{array}{ccccccccccc}
1 & + & A_8 & + & A_{16} & + & A_{24} & + & A_{32} & = & 8y, \\
52 & + & 44A_8 & + & 36A_{16} & + & 28A_{24} & + & 20A_{32} & = & 4y \cdot 52, \\
\binom{52}{2} & + & \binom{44}{2}A_8 & + & \binom{36}{2}A_{16} & + & \binom{28}{2}A_{24} & + & \binom{20}{2}A_{32} & = & 2y \cdot \binom{52}{2}, \\
\binom{52}{3} & + & \binom{44}{3}A_8 & + & \binom{36}{3}A_{16} & + & \binom{28}{3}A_{24} & + & \binom{20}{3}A_{32} & = & y\left(\binom{52}{3} + A_3^\perp\right)
\end{array}$$

are the first 4 MacWilliams Identities using $A_{40} = A_{48} = 0$ from a recursive application of the linear programming method, where $y = 2^{v-3}$.

Linear programming method

Example (cont.)

Substituting $x = yA_3^{\perp}$ and solving for A_8, A_{16}, A_{24}, A_{32} yields

$A_8 = -4 + \frac{1}{512}x + \frac{7}{64}y$, $A_{16} = 6 - \frac{3}{512}x - \frac{17}{64}y$,
$A_{24} = -4 + \frac{3}{512}x + \frac{397}{64}y$, and $A_{32} = 1 - \frac{1}{512}x + \frac{125}{64}y$.

Since $A_{16}, x \geq 0$, we have $y \leq \frac{384}{17} < 23$. On the other hand, since $3A_8 + A_{16} \geq 0$, we also have $-6 + \frac{y}{16} \geq 0$, i.e., $y \geq 96$ – a contradiction.

First 2 MacWilliams identities

Definition

For a point set $\mathcal{C}$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ let $\mathcal{T}(\mathcal{C}) := \{0 \le i \le c \mid a_i > 0\}$, where a_i denotes the number of hyperplanes with $\#(\mathcal{C} \cap H) = i$.

Lemma

For integers $u \in \mathbb{Z}$, $m \ge 0$ and $\Delta \ge 1$ let $\mathcal{C}$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ be Δ-divisible of cardinality $n = u + m\Delta \ge 0$. Then, we have $(q-1) \cdot \sum_{h\in\mathbb{Z},h\le m} h a_{u+h\Delta} = (u + m\Delta - uq) \cdot \frac{q^{v-1}}{\Delta} - m$, where we set $a_{u+h\Delta} = 0$ if $u + h\Delta < 0$.

Corollary

Let $\mathcal{C}$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ satisfy $n = \#\mathcal{C} = u + m\Delta$ and $\mathcal{T}(\mathcal{C}) \subseteq \{u, u+\Delta, \ldots, u+m\Delta\}$. Then $u < \frac{n}{q}$ or $u = n = 0$.

First 2 MacWilliams identities

Applied recursively, we obtain:

Theorem Năstase and Sissokho 2016

Suppose $v = tk + r$ with $t \geq 1$ and $0 < r < k$. If $k > \frac{q^r-1}{q-1}$ then $A_q(v, 2k; k) = 1 + \sum_{i=1}^{t-1} q^{ik+r} = \frac{q^v - q^{k+r} + q^k - 1}{q^k - 1}$.

Remark

If k is *large*, then the construction of Beutelspacher is optimal.

Remark

We have utilized the non-negativity of a certain linear polynomial (in a given range) it the stated Lemma.

First 3 MacWilliams identities

Lemma

Let $\Delta = q^{s-1}$, $m \in \mathbb{Z}$, and $\mathcal{P}$ be a partial s-spread in $\mathbb{F}_q^v$ with c holes. Then, $\tau_q(c, \Delta, m) \cdot \frac{q^{v-2}}{\Delta^2} - m(m-1) \geq 0$ and $\tau_q(c, \Delta, m) \geq 0$, where $\tau_q(c, \Delta, m) =$
$m(m-1)\Delta^2 q^2 - c(2m-1)(q-1)\Delta q + c(q-1)\Big(c(q-1)+1\Big)$.
If $c > 0$, then $\tau_q(c, \Delta, m) = 0$ if and only if $m = 1$ and $c = \frac{q^s-1}{q-1}$.

Theorem K. 2016

For integers $r \geq 1$, $t \geq 2$, $u \geq 0$, and $0 \leq z \leq \frac{q^r-1}{q-1}/2$ with $k = \frac{q^r-1}{q-1} + 1 - z + u > r$ we have $A_q(v, 2k; k) \leq lq^k + 1 + z(q-1)$, where $l = \frac{q^{v-k}-q^r}{q^k-1}$ and $v = kt + r$.

First 3 MacWilliams identities

Theorem K. 2016

For integers $r \geq 1$, $t \geq 2$, $y \geq \max\{r, 2\}$, $z \geq 0$ with $\lambda = q^y$, $y \leq k$, $k = \frac{q^r-1}{q-1} + 1 - z > r$, $v = kt + r$, and $l = \frac{q^{v-k}-q^r}{q^k-1}$, we have $A_q(v, 2k; k) \leq$
$lq^k + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\lambda\left(\lambda - (z + y - 1)(q - 1) - 1\right)} \right\rceil$.

Remark

Setting $y = k$, we obtain the bound of Drake and Freeman.

Remark

We have utilized the non-negativity of a certain quadratic polynomial it the stated Lemma.

First 4 MacWilliams identities

Lemma K. 2016

Let $\mathcal{C}$ be Δ-divisible over $\mathbb{F}_q$ of cardinality $n > 0$ and $t \in \mathbb{Z}$. Then $\sum_{i \ge 1} \Delta^2 (i-t)(i-t-1) \cdot (g_1 \cdot i + g_0) \cdot A_{i\Delta} + qhx = n(q-1)(n-t\Delta)(n-(t+1)\Delta) g_2$, where $g_1 = \Delta q h$, $g_0 = -n(q-1) g_2$, $g_2 = h - (2\Delta q t + \Delta q - 2nq + 2n + q - 2)$ and $h = \Delta^2 q^2 t^2 + \Delta^2 q^2 t - 2\Delta n q^2 t - \Delta n q^2 + 2\Delta n q t + n^2 q^2 + \Delta n q - 2n^2 q + n^2 + nq - n$.

Corollary

If there exists $t \in \mathbb{Z}$, using the above notation, with $n/\Delta \notin [t, t+1]$, $h \ge 0$, and $g_2 < 0$, then there is no Δ-divisible set over $\mathbb{F}_q$ of cardinality n.

First 4 MacWilliams identities

Remark

We have utilized the non-negativity of a certain cubic polynomial it the stated Lemma.

- $2^4 l + 1 \le A_2(4k+3, 8; 4) \le 2^4 l + 4$, where $l = \frac{2^{4k-1} - 2^3}{2^4 - 1}$;
- $2^6 l + 1 \le A_2(6k+4, 12; 6) \le 2^6 l + 8$, where $l = \frac{2^{6k-2} - 2^4}{2^6 - 1}$;
- $2^6 l + 1 \le A_2(6k+5, 12; 6) \le 2^6 l + 18$, where $l = \frac{2^{6k-1} - 2^5}{2^6 - 1}$;
- $3^4 l + 1 \le A_3(4k+3, 8; 4) \le 3^4 l + 14$, where $l = \frac{3^{4k-1} - 3^3}{3^4 - 1}$;
- $3^5 l + 1 \le A_3(5k+3, 10; 5) \le 3^5 l + 13$, where $l = \frac{3^{5k-2} - 3^5}{3^3 - 1}$;
- $3^5 l + 1 \le A_3(5k+4, 10; 5) \le 3^5 l + 44$, where $l = \frac{3^{5k-1} - 3^4}{3^5 - 1}$;
- $3^6 l + 1 \le A_3(6k+4, 12; 6) \le 3^6 l + 41$, where $l = \frac{3^{6k-2} - 3^4}{3^6 - 1}$;
- $3^6 l + 1 \le A_3(6k+5, 12; 6) \le 3^6 l + 133$, where $l = \frac{3^{6k-1} - 3^5}{3^6 - 1}$;
- $3^7 l + 1 \le A_3(7k+4, 14; 7) \le 3^7 l + 40$, where $l = \frac{3^{7k-3} - 3^4}{3^7 - 1}$;

First 4 MacWilliams identities

- $4^5 l + 1 \leq A_4(5k+3, 10; 5) \leq 4^5 l + 32$, where $l = \frac{4^{5k-2} - 4^3}{4^5 - 1}$;
- $4^6 l + 1 \leq A_4(6k+3, 12; 6) \leq 4^6 l + 30$, where $l = \frac{4^{6k-3} - 4^3}{4^6 - 1}$;
- $4^6 l + 1 \leq A_4(6k+5, 12; 6) \leq 4^6 l + 548$, where $l = \frac{4^{6k-1} - 4^5}{4^6 - 1}$;
- $4^7 l + 1 \leq A_4(7k+4, 14; 7) \leq 4^7 l + 128$, where $l = \frac{4^{7k-3} - 4^4}{4^7 - 1}$;
- $5^5 l + 1 \leq A_5(5k+2, 10; 5) \leq 5^5 l + 7$, where $l = \frac{5^{5k-3} - 5^2}{5^5 - 1}$;
- $5^5 l + 1 \leq A_5(5k+4, 10; 5) \leq 5^5 l + 329$, where $l = \frac{5^{5k-1} - 5^4}{5^5 - 1}$;
- $7^5 l + 1 \leq A_7(5k+4, 10; 5) \leq 7^5 l + 1246$, where $l = \frac{7^{5k-1} - 7^2}{7^5 - 1}$;
- $8^4 l + 1 \leq A_8(4k+3, 8; 4) \leq 8^4 l + 264$, where $l = \frac{8^{4k-1} - 8^3}{8^4 - 1}$;
- $8^5 l + 1 \leq A_8(5k+2, 10; 5) \leq 8^5 l + 25$, where $l = \frac{8^{5k-3} - 8^2}{8^5 - 1}$;
- $8^6 l + 1 \leq A_8(6k+2, 12; 6) \leq 8^6 l + 21$, where $l = \frac{8^{6k-4} - 8^2}{8^6 - 1}$;
- $9^3 l + 1 \leq A_9(3k+2, 6; 3) \leq 9^3 l + 41$, where $l = \frac{9^{3k-1} - 9^2}{9^3 - 1}$;
- $9^5 l + 1 \leq A_9(5k+3, 10; 5) \leq 9^5 l + 365$, where $l = \frac{9^{5k-2} - 9^3}{9^5 - 1}$.

What have we done?

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Determined a feasible solution for the dual of the linear program corresponding to the first r MacWilliams identities for $2 \leq r \leq 4$ using certain polynomials fo degree $r - 1$.

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Open problem

What about the first $r = 5$ MacWilliams identities?
How does a suitable quartic polynomial look like?

Visit us

Table for $A_2(13, d; k)$

d\k	2	3	4	5	6
4	**2729**	**1597245**	157319501 - 217544769	4794061075 - 7193022828	38325127529 - 57886442918
6		**1169**	266891 - 319449	16835124 - 20918757	269057345 - 339835228
8			**545**	65793 - 72133	2097225 - 2284118
10				257 - 260	16385 - 16772
12					**129**

http://subspacecodes.uni-bayreuth.de/

Thank you very much for your attention!