# Gröbner Bases over Finitely Generated Affine Monoids and Applications

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Kassel

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# Outline

- Cauchy problem for 1d systems
- Finitely generated affine monoids
- Gröbner bases for submonoid algebras
- Cauchy problem for $n$d systems
- Computation of Gröbner bases
- Construction of generalised term orders

# Cauchy Problem for 1d Discrete Systems

**1 equation, 1 unknown:**

$$a_d w(t+d) + a_{d-1} w(t+d-1) + \cdots + a_0 w(t) = v(t), \quad t \in \mathbb{N}$$

$$\left.\begin{array}{c} a_d, \ldots, a_0 \in F \text{ (field)},\ \ a_d \neq 0 \\ v = (v(t))_{t\in\mathbb{N}} \in F^{\mathbb{N}} \end{array}\right\} \text{ (known)},$$

$$\text{solution } w = (w(t))_{t\in\mathbb{N}} \in F^{\mathbb{N}} \text{ (unknown)}.$$

**Computation of $w$:**

$$w(t+d) = -\frac{a_{d-1}}{a_d} w(t+d-1) - \cdots - \frac{a_0}{a_d} w(t) + \frac{1}{a_d} v(t).$$

- choose $w(d-1), \ldots, w(0) \in F$ freely,
- compute $w(t)$ recursively.

# Computation of $w$ – Alternative Method

$$a_d w(t+d) + a_{d-1} w(t+d-1) + \cdots + a_0 w(t) = v(t), \quad t \in \mathbb{N}.$$

$f := a_d s^d + a_{d-1} s^{d-1} + \cdots + a_0 \in F[s]$.
Scalar multiplication

$$\circ\colon\ F[s] \times F^{\mathbb{N}} \longrightarrow F^{\mathbb{N}}, \quad (s \circ w)(t) := w(t+1) \quad \text{(left shift)}.$$

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$w(0), \ldots, w(d-1)$ and $v \in F^{\mathbb{N}}$ given.

$$\begin{aligned} w(t) &= (s^t \circ w)(0) = \big((qf+r) \circ w\big)(0) \\ &= \big(q \circ (\underbrace{f \circ w}_{=v})\big)(0) + (r \circ w)(0) = (q \circ v)(0) + (r \circ w)(0) \\ &= \underbrace{q_n v(n) + \cdots + q_0 v(0)}_{\text{inhomogeneity}} + \underbrace{r_{d-1} w(d-1) + \cdots + r_0 w(0)}_{\text{initial values}} \end{aligned}$$

Division with remainder: $s^t = qf + r$
with $q = \sum_{i=0}^{n} q_i s^i$, $r = \sum_{j=0}^{d-1} r_j s^j$.

# Cauchy Problem for 1d Discrete Systems

$k$ **equations, 1 unknown:**

$$\left.\begin{array}{c} f_1 \circ w = v_1 \\ \vdots \\ f_k \circ w = v_k \end{array}\right\} \iff \gcd(f_1, \ldots, f_k) \circ w = v'.$$

$k$ **equations,** $l$ **unknowns:** $R \circ \begin{pmatrix} w_1 \\ \vdots \\ w_l \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$

with $R \in F[s]^{k \times l}$ matrix of difference operators.

Solve Cauchy problem via Smith form or Hermite form.

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**Relevant algebraic objects:**

$$F[s]f_1 + \cdots + F[s]f_k = F[s]\gcd(f_1, \ldots, f_k) \subseteq F[s] \quad \text{ideal.}$$

$$F[s]R_{1-} + \cdots + F[s]R_{k-} = F[s]^{1\times k}R \subseteq F[s]^{1\times l} \qquad \text{row module.}$$

# Multidimensional Systems

| | 1d | nd |
|---|---|---|
| operators | $\sum_{i\in\mathbb{N}} a_i s^i$ $\in F[s]$ | $\sum_{\mu\in\mathbb{N}^n} a_\mu s^\mu$ $\in F[s_1,\ldots,s_n] = F[s]$ |
| tools | gcd, Smith form<br>division with remainder | Gröbner bases<br>division with remainder (normal form) |
| solutions | $w = (w(t))_{t\in\mathbb{N}} \in F^{\mathbb{N}}$ | $w = (w(\mu))_{\mu\in\mathbb{N}^n} \in F^{\mathbb{N}^n}$<br>$\mathbb{N}^n$ ... positive orthant |

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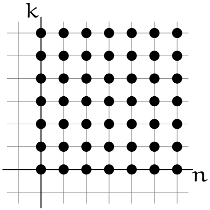
# Example: Binomial Coefficients

$\mu = (n, k)$, $w(\mu) = \binom{n}{k}$,
$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k} \iff f \circ w = 0$ with $f = s_1 s_2 - s_2 - 1$.

**naive method**

$f \in \mathbb{R}[s_1, s_2]$,
$w \in \mathbb{R}^{\mathbb{N}^2}$ positive orthant.




Initial values:
1d: $\{0, \ldots, \deg(f) - 1\}$
$= \mathbb{N} \setminus \deg(F[s]f)$,
nd: $\mathbb{N}^n \setminus \deg(F[s_1, \ldots, s_n]f)$.

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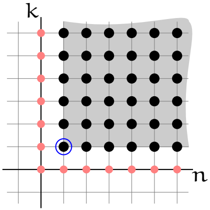


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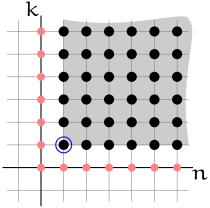


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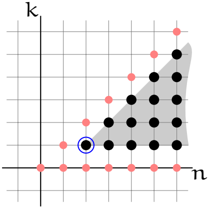
**usually**

Initial values: $\binom{n}{0} = \binom{n}{n} = 1$,
$n \in \mathbb{N}$.
Interested in $\binom{n}{k}$ for $k \in \{0, \ldots, n\}$.




$w \in \mathbb{R}^N$, $N := \mathbb{N}(1,0) + \mathbb{N}(1,1)$,
$s_1 f = s_1^2 s_2 + s_1 s_2 + s_1 \in \mathbb{R}[s_1, s_1 s_2]$.

# Finitely Generated Affine Monoids

$N \subseteq \mathbb{Z}^{1\times n}$ is affine monoid $:\Longleftrightarrow$

- $0 \in N$,
- $N$ closed under $+$.

Affine monoid $N$ is finitely generated $:\Longleftrightarrow$

$$\exists \theta \in \mathbb{Z}^{m\times n}\colon\ N = \mathbb{N}^{1\times m}\theta = \textstyle\sum_{i=1}^{m} \mathbb{N}\theta_{i-}.$$

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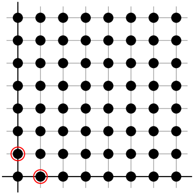
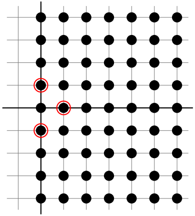
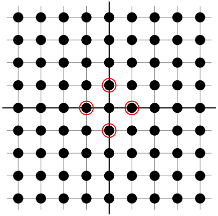


$N = \mathbb{N}^{1\times 2}$



$N = \mathbb{N} \times \mathbb{Z}$



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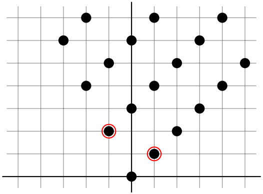
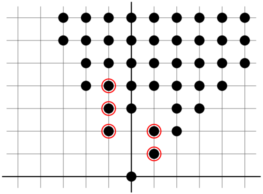
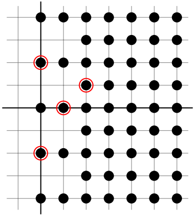


$$N = \mathbb{N}^{1\times 2}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$



$$N = \mathbb{N}^{1\times 5}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 2 \\ -1 & 3 \\ -1 & 4 \end{pmatrix}$$



$$N = \mathbb{N}^{1\times 4}\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \\ 0 & -2 \end{pmatrix}$$

# Monoid Algebras

Field $F$.

Solution space $F^N \ni w = (w(\mu))_{\mu \in N}$.

Monoid algebra

$$F[N] := \left\{ \sum_{\mu \in N} p_\mu \sigma^\mu ;\ p_\mu \in F,\ \text{only finitely many } \neq 0 \right\}$$

with $\left(\sum_{\mu \in N} p_\mu \sigma^\mu\right)\left(\sum_{\mu \in N} q_\mu \sigma^\mu\right) = \sum_{\mu,\nu \in N} p_\mu q_\nu \sigma^{\mu+\nu}$.

Scalar product $(s^\mu \circ w)(\nu) = w(\nu + \mu), \quad \mu, \nu \in N,\ w \in F^N$.

**Binomial coefficients:**

$N = \mathbb{N}(1,0) + \mathbb{N}(1,1) = \mathbb{N}^{1\times 2}\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \subseteq \mathbb{Z}^{1\times 2}$ f.g. affine monoid.
Solution $w \in \mathbb{R}^N$.
Operator $s_1 f = s_1^2 s_2 + s_1 s_2 + s_1 \in \mathbb{R}[s_1, s_1 s_2] = \mathbb{R}[N]$.

# Goal

Gröbner bases for submodules of $\mathbb{F}[N]^{1\times l}$ in analogy to
Gröbner bases for submodules of $\mathbb{F}[\sigma]^{1\times l}$.

In this talk: GB for ideals in $\mathbb{F}[N]$ $\leftrightsquigarrow$ $k$ equations, 1 unknown.

# Gröbner Bases

$N = \mathbb{N}^{1\times n}$, $F[N] = F[\sigma] = F[\sigma_1, \ldots, \sigma_n]$,

$\leqslant$ ... term order on $\mathbb{N}^{1\times n}$, i.e.,

- total order,
- $\min \mathbb{N}^{1\times n} = 0$,
- group order, i.e., $\mu \leqslant \nu \implies \mu + \eta \leqslant \nu + \eta$ for $\mu, \nu, \eta \in \mathbb{N}^{1\times n}$.

Then:

- $\deg(p) = \max\{\mu;\ p_\mu \neq 0\}$ where $p = \sum_{\mu\in\mathbb{N}^{1\times n}} p_\mu \sigma^\mu \in F[\sigma]$,
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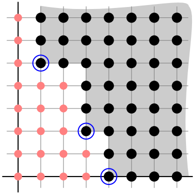
$$\deg(\mathfrak{a}) = \bigcup_{g\in\mathcal{G}} \left(\deg(g) + \mathbb{N}^{1\times n}\right).$$



$\bullet \notin \deg(\mathfrak{a})$

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**Theorem:** $\leqslant$ term order on $\mathbb{N}^{1\times n}$.

$$\begin{aligned} F[\sigma] &= F^{\left(\mathbb{N}^{1\times n}\setminus\deg(\mathfrak{a})\right)} \oplus \mathfrak{a} \\ p &= \qquad p_{\mathrm{nf}} \qquad + (p - p_{\mathrm{nf}}) \end{aligned}$$

$p_{\mathrm{nf}}$ ... normal form of $p$, computed via Gröbner bases.

# Conic Decompositions

$\mathrm{N} \subseteq \mathbb{Z}^{1\times n}$ ... arbitrary f.g. affine monoid.

**Problem:** If $\mathrm{N}$ is not pointed, e.g., $\mathrm{N} = \mathbb{Z} = \mathbb{N} \cdot 1 + \mathbb{N} \cdot (-1)$, then there are no total group orders on $\mathrm{N}$ with $0 = \min \mathrm{N}$.

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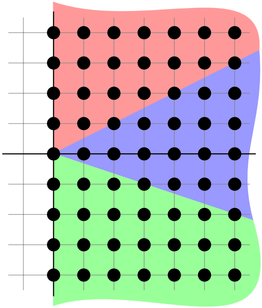
**Definition:** Finite family $(\mathrm{N}_J)_{J\in\mathcal{J}}$ is conic decomposition of $\mathrm{N}$ :$\Longleftrightarrow$

- $\forall J \in \mathcal{J}$: $\mathrm{N}_J$ is pointed f.g. submonoid of $\mathbb{Z}^{1\times n}$,
- $\forall J \in \mathcal{J}$: $\mathbb{Z}\mathrm{N}_J = \mathbb{Z}\mathrm{N}$, where $\mathbb{Z}\mathrm{N} = \mathrm{N} - \mathrm{N}$,
- $\bigcup_{J\in\mathcal{J}} \mathrm{N}_J = \mathrm{N}$.

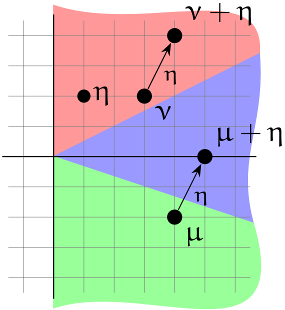


# Generalised Term Orders

**Definition:** $\leqslant$ is a generalised term order on $\mathbb{N}$ w.r.t. a conic decomposition $(\mathbb{N}_J)_{J\in\mathcal{J}}$ $:\Longleftrightarrow$

- total order,
- $0 = \min \mathbb{N}$,
- $\forall J \in \mathcal{J}\ \forall \mu \in \mathbb{N}\ \forall \nu, \eta \in \mathbb{N}_J:$

$$\mu \leqslant \nu \Longrightarrow \mu + \eta \leqslant \nu + \eta.$$



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- $\deg(\mathfrak{a}) = \{\deg(p);\ 0 \neq p \in \mathfrak{a}\}$ for $\mathfrak{a} \subseteq F[N]$ ideal.
- $\deg(ps^\nu) = \deg(p) + \nu$ if $\exists J \in \mathcal{J}$ with $\deg(p), \nu \in N_J$.

# Generalised Gröbner Bases

$N \subseteq \mathbb{Z}^{1\times n}$ f.g. affine monoid, $F[N] \subseteq F[\sigma, \sigma^{-1}]$ monoid algebra, $(N_J)_{J\in\mathcal{J}}$ conic decomposition of $N$, $\leqslant$ generalised term order on $N$.

**Definition:** $\mathcal{G} \subseteq F[N]$ Gröbner basis of ideal $\mathfrak{a} \subseteq F[N]$ $:\Longleftrightarrow$
$\mathcal{G}$ generates $\mathfrak{a}$ and

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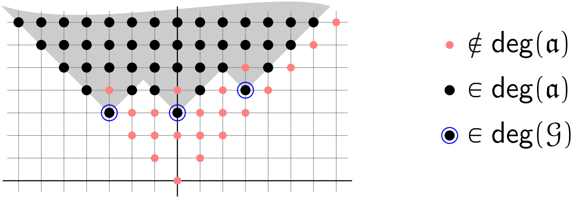

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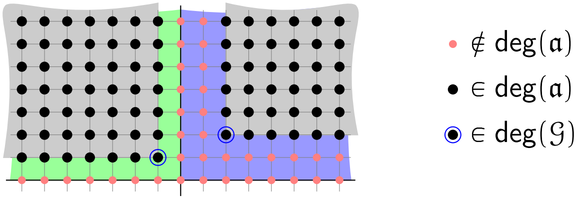

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**Theorem:** $(N_J)_{J\in\mathcal{J}}$ conic decomposition, $\leqslant$ generalised term order on $\mathbb{N}^{1\times n}$.

$$\begin{aligned} F[N] &= F^{\left(N\setminus\deg(\mathfrak{a})\right)} \oplus \mathfrak{a} \\ p &= \qquad p_{\mathrm{nf}} \qquad + (p - p_{\mathrm{nf}}) \end{aligned}$$

$p_{\mathrm{nf}}$ ... normal form of $p$, computed via Gröbner bases.

# Cauchy Problem for $\mathfrak{n}$d Discrete Systems

$$R \circ w = v, \qquad R \in F[N]^{k \times 1}, \quad v \in (F^N)^k, \quad \mu \in N, \quad w(\mu) = ?$$

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$$R \circ w = v, \qquad R \in F[N]^{k\times 1}, \quad v \in (F^N)^k, \quad \mu \in N, \quad w(\mu) =?$$

$\mathfrak{a} := F[N]^{1\times k}R$. Let $P \in F[N]^{1\times k}$ such that

$$s^\mu = (s^\mu)_{\mathrm{nf}} + PR \in F^{\left(N\setminus \deg(\mathfrak{a})\right)} \oplus \mathfrak{a}.$$

Then:

$$\begin{aligned} w(\mu) &= (s^\mu \circ w)(0) = \big((s^\mu)_{\mathrm{nf}} \circ w\big)(0) + \big(P \circ (\underbrace{R \circ w}_{=v})\big)(0) \\ &= \big((s^\mu)_{\mathrm{nf}} \circ w\big)(0) + (P \circ v)(0) \\ &= \underbrace{\sum_{\nu \in N\setminus \deg(\mathfrak{a})} \text{coeff } w(\nu)}_{\text{initial values}} + \underbrace{\sum_{i=1}^{k} \sum_{\eta\in N} \text{coeff } v_i(\eta)}_{\text{inhomogeneity}} \end{aligned}$$

$\Longrightarrow N\setminus \deg(\mathfrak{a})$ ... initial value region.

# Cauchy Problem for $n$d Discrete Systems

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$\Longrightarrow N\setminus\deg(\mathfrak{a})$ ... initial value region.

**Theorem:** Assumption: Solution $w$ exists. Then:

$$\forall x \in F^{N\setminus\deg(\mathfrak{a})}\ \exists_1 w \in F^N:\ \ R \circ w = v \text{ and } w|_{N\setminus\deg(\mathfrak{a})} = x.$$

# Literature

- F. Pauer and S. Zampieri. Gröbner bases with respect to generalized term orders and their application to the modelling problem. J. Symb. Comput., 21(2):155–168, 1996.
- F. Pauer and A. Unterkircher. Gröbner bases for ideals in Laurent polynomial rings and their application to systems of difference equations. Appl. Algebra Engrg. Comm. Comput., 9(4):271–291, 1999.

**Results:**

- Gröbner bases for modules over $\mathrm{F}[\mathrm{N}]$ for arbitrary f.g. affine monoids $\mathrm{N}$,
- Algorithm for their computation (generalised Buchberger algorithm).

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**Results:**

- Gröbner bases for modules over $\mathbb{F}[N]$ for arbitrary f.g. affine monoids $N$,
- Algorithm for their computation (generalised Buchberger algorithm).

**Disadvantages:**

- Algorithm works directly in $\mathbb{F}[N] \rightsquigarrow$ complete re-implementation of Buchberger algorithm necessary.
- Existence of conic decompositions and generalised term orders for arbitrary f.g. affine monoids $N$ not shown.

# New Algorithm – Idea

$N = \mathbb{N}^{1\times m}\theta$, $\theta \in \mathbb{Z}^{m\times n}$.

$$\psi\colon \mathbb{N}^{1\times m} \longrightarrow N, \quad \tilde{\mu} \longmapsto \tilde{\mu}\theta,$$

surjective, i.e., parametrisation of $N$.

$$\varphi\colon F[s] = F[s_1, \ldots, s_m] = F[\mathbb{N}^{1\times m}] \longrightarrow F[N], \quad s^{\tilde{\mu}} \longmapsto \sigma^{\tilde{\mu}\theta},$$

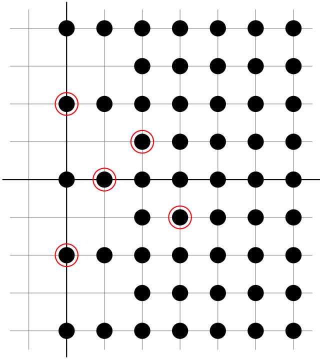
algebra epimorphism, parametrisation of $F[N]$.

**Idea:** Compute GB in $F[s]$, transport result to $F[N]$.

**Necessary:** Compatible data $\theta \in \mathbb{Z}^{m\times n}$, $\mathcal{J} \subseteq \mathcal{P}(1, \ldots, m)$, $\preccurlyeq$ term order on $\mathbb{N}^{1\times m}$ which induces a conic decomposition and a generalised term order on $N$.

# Construction of Generalised Term Orders

Given: $\theta' \in \mathbb{Z}^{m' \times n}$ with $\mathbb{N}^{1 \times m'}\theta' = N$.

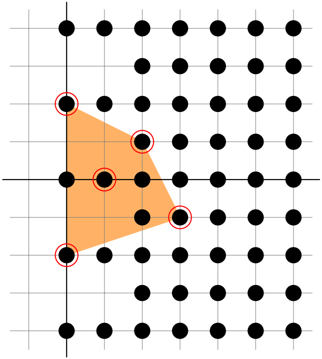


# Construction of Generalised Term Orders

Given: $\theta' \in \mathbb{Z}^{m' \times n}$ with $\mathbb{N}^{1 \times m'}\theta' = N$.

Define

- $P := \mathrm{conv}(\theta'_{1-}, \ldots, \theta'_{m'-}, 0) \subseteq \mathbb{Q}^{1 \times n}$ polytope,



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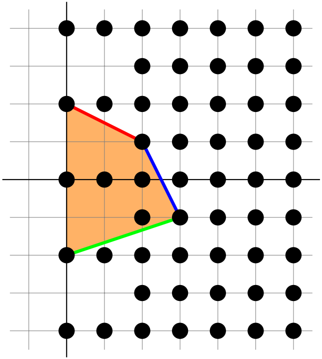

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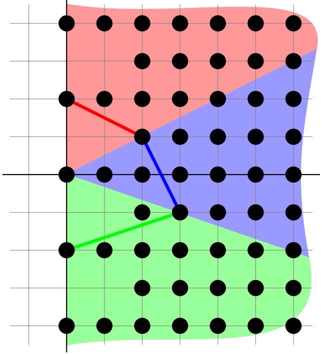
- $\mathcal{F} := \{F \subseteq P \text{ facet with } 0 \notin F\}$,
- $N_F := \mathbb{Q}_{\geqslant 0} F \cap N$, where $\mathbb{Q}_{\geqslant 0} F$ … convex rational cone generated by $F$.

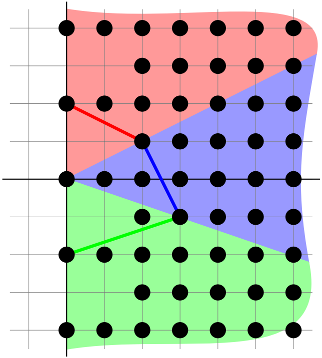
Then: $N_F$ f.g. submonoid, $(N_F)_{F \in \mathcal{F}}$ conic decomposition of $N$.



# Construction of Generalised Term Orders

$F \in \mathcal{F}$. Let $f_F \colon \mathbb{Q}N \longrightarrow \mathbb{Q}$ be linear with $f_F(P) \leqslant 1$ and $f_F(F) = 1$.

Define $f \colon \mathbb{Q}N \longrightarrow \mathbb{Q}$, $\mu \longmapsto \max\left\{f_F(\mu);\ F \in \mathcal{F}\right\}$.



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For $\mu, \nu \in N$ define

$$\begin{aligned} \mu \leqslant \nu :\Longleftrightarrow\ & f(\mu) < f(\nu) \\ & \text{or } \Big(f(\mu) = f(\nu) \text{ and } \mu \leqslant_{\mathbb{Z}^{1\times n}} \nu\Big), \end{aligned}$$

where $\leqslant_{\mathbb{Z}^{1\times n}}$ is a group order on $\mathbb{Z}^{1\times n} \supseteq N$.

**Theorem:** $\leqslant$ is a generalised term order w.r.t. the conic decomposition $(N_F)_{F\in\mathcal{F}}$.

# Construction of Compatible Data

$N_F = \mathbb{Q}_{\geqslant 0} F \cap N$ is f.g. submonoid.
Let $\theta_F \in \mathbb{Z}^{\bullet \times n}$ with $N_F = \mathbb{Z}^{1\times\bullet}\theta_F$.

$$\theta := \begin{pmatrix} \theta_{F_1} \\ \vdots \\ \theta_{F_r} \end{pmatrix} \in \mathbb{Z}^{m\times n},\ \ F_i \in \mathcal{F},$$

$$J_F := \{i;\ \ \theta_{i-} \in N_F\},$$

$$\mathcal{J} := \{J_F;\ \ F \in \mathcal{F}\} \subseteq \mathcal{P}(1,\ldots,m)$$

$$\tilde{f} \colon \mathbb{N}^{1\times m} \longrightarrow \mathbb{Q}, \quad \tilde{\mu} \longmapsto \textstyle\sum_{i=1}^{m} \tilde{\mu}_i f(\theta_{i-})$$

For $\tilde{\mu}, \tilde{\nu} \in \mathbb{N}^{1\times m}$ define

$$\begin{aligned} \tilde{\mu} \preccurlyeq \tilde{\nu} :\Longleftrightarrow\ & \tilde{f}(\tilde{\mu}) < \tilde{f}(\tilde{\nu}) \\ & \text{or } \left(\tilde{f}(\tilde{\mu}) = \tilde{f}(\tilde{\nu}) \text{ and } \tilde{\mu}\theta <_{\mathbb{Z}N} \tilde{\nu}\theta\right) \\ & \text{or } \left(\tilde{f}(\tilde{\mu}) = \tilde{f}(\tilde{\nu}) \text{ and } \tilde{\mu}\theta = \tilde{\nu}\theta \text{ and } \tilde{\mu} \preccurlyeq_1 \tilde{\nu}\right), \end{aligned}$$

where $\leqslant_{\mathbb{Z}^{1\times n}}$ is a group order on $\mathbb{Z}^{1\times N}$, $\preccurlyeq_1$ term order on $\mathbb{N}^{1\times m}$.

**Theorem:** $(\theta, \mathcal{J}, \preccurlyeq)$ are compatible data for $N$.

# Summary

For f.g. submonoids N, we have

- a way to construct generalised term orders and compatible data,
- a Gröbner basis theory,
- an efficient algorithm for the computation of these Gröbner bases, which required for
- an algorithm for solving the Cauchy problem.

M. Scheicher. Gröbner bases and their application to the Cauchy problem on finitely generated affine monoids. *J. Symb. Comput.* 80(2): 416–450, 2017.

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**Future work:**

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- Provide a library in SINGULAR.
- Classify all generalised term orders (Robbiano's theorem).

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**Thank you!**