Asymptotically Fast Arithmetic in the Picard Group of Algebraic Curves

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Computeralgebra -Tagung Kassel

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- Reduce arithmetic in $\mathrm{Pic}^0(C)$ to linear algebra over $k[x]$ in
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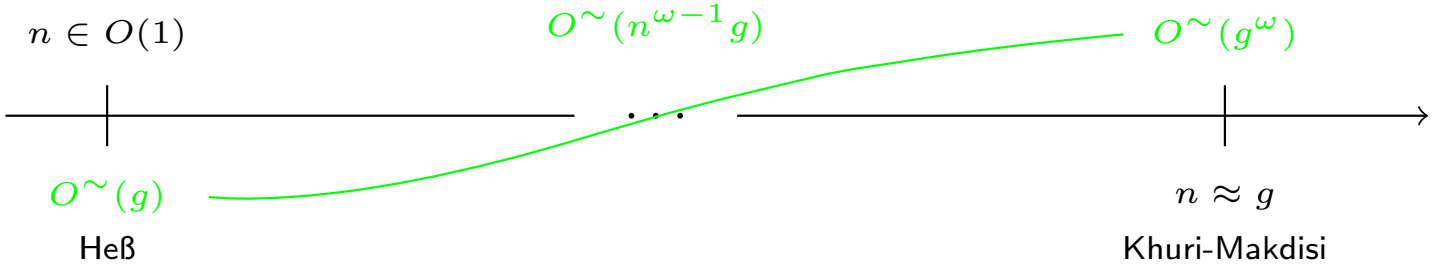

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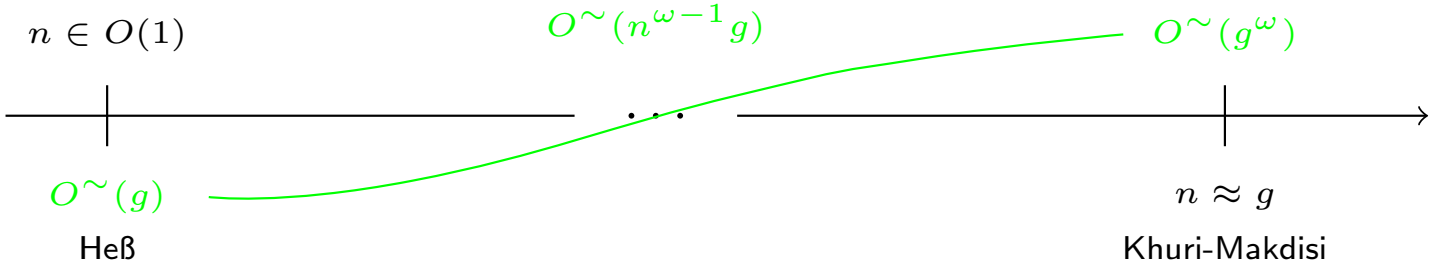

(2) Generalize to the **singular** case where

there are **no results** until today!

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- Those sections form invertible ideals of the coordinate rings

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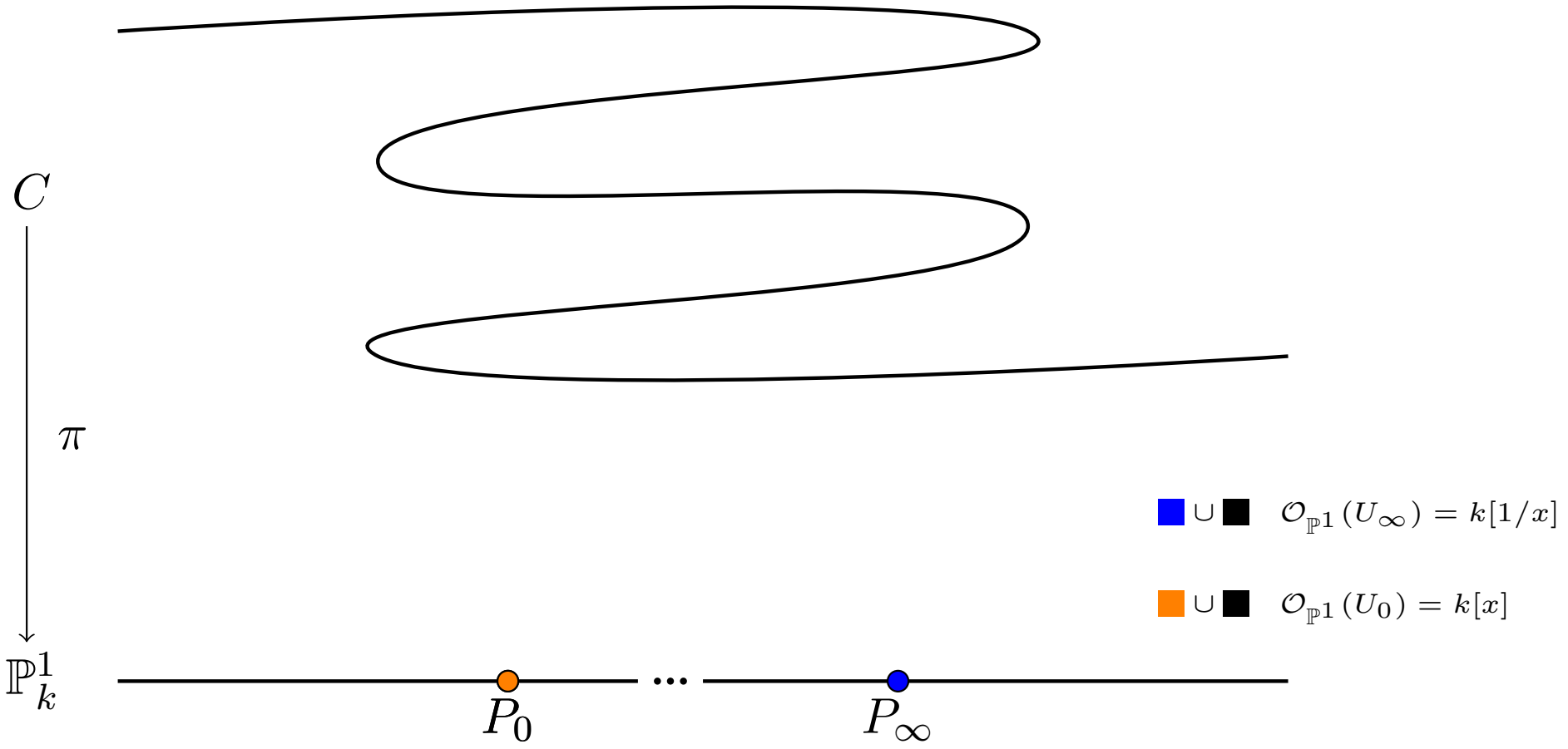
- C integral projective curve over the field k
- $\pi : C \to \mathbb{P}^1_k$ finite morphism of degree n which is separable



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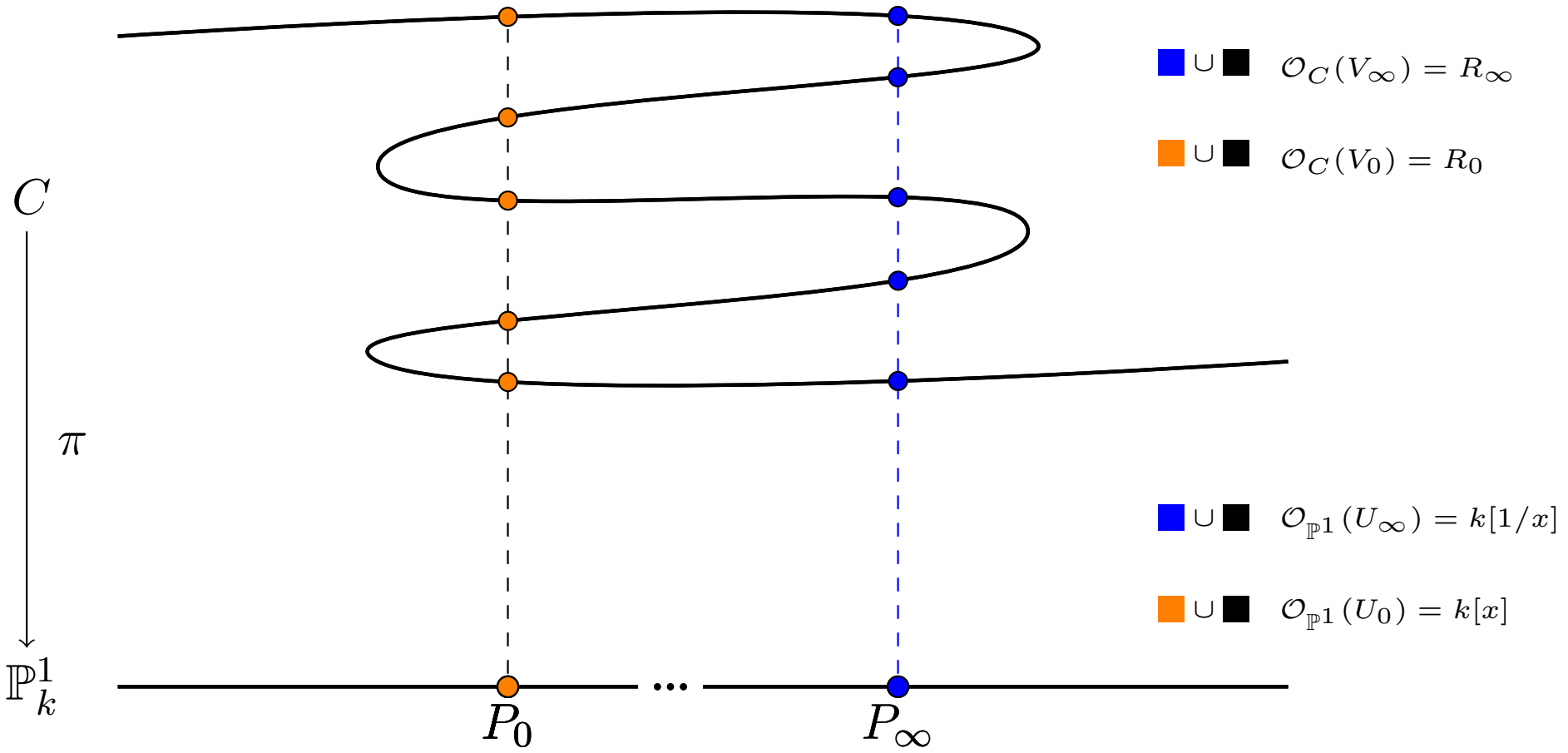

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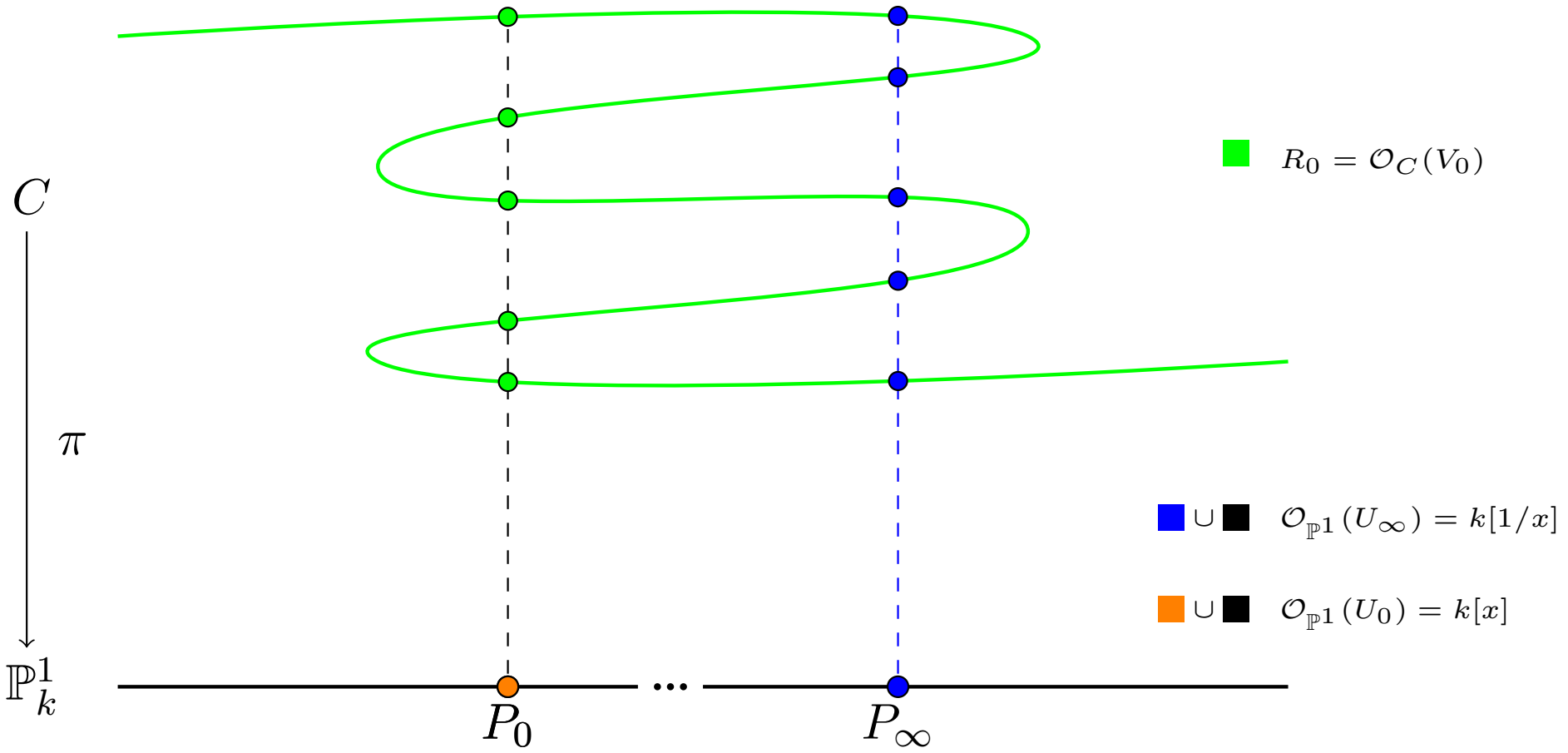

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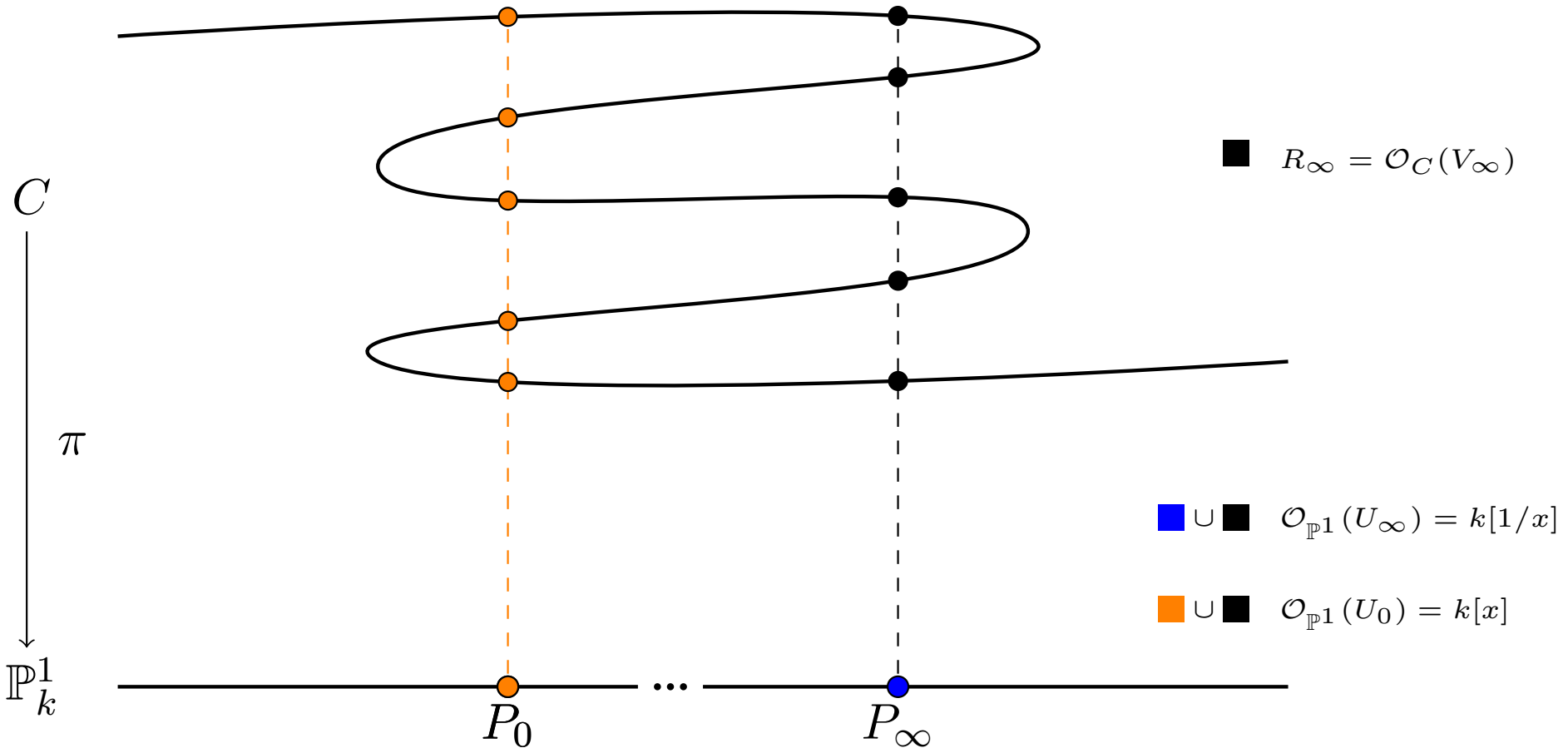

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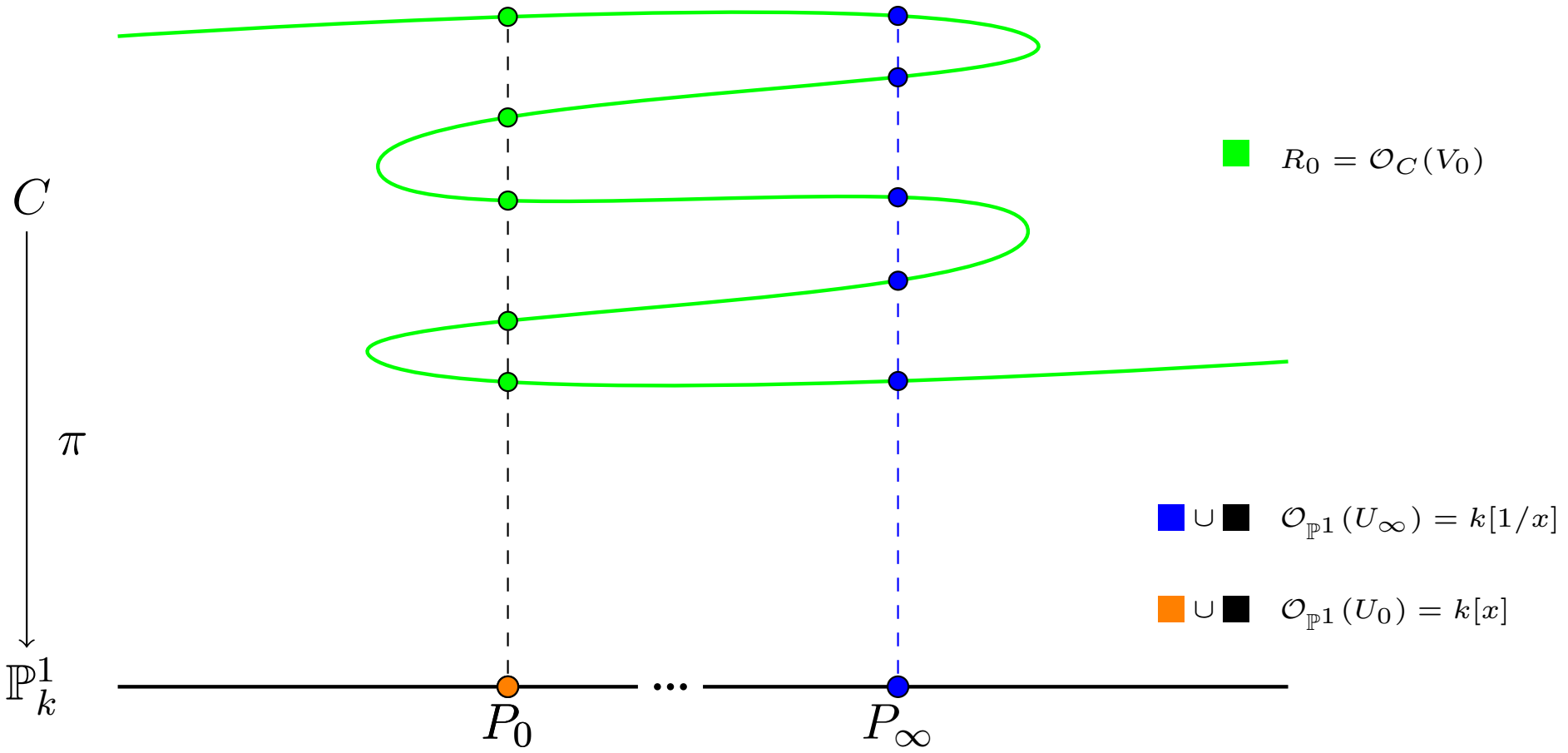

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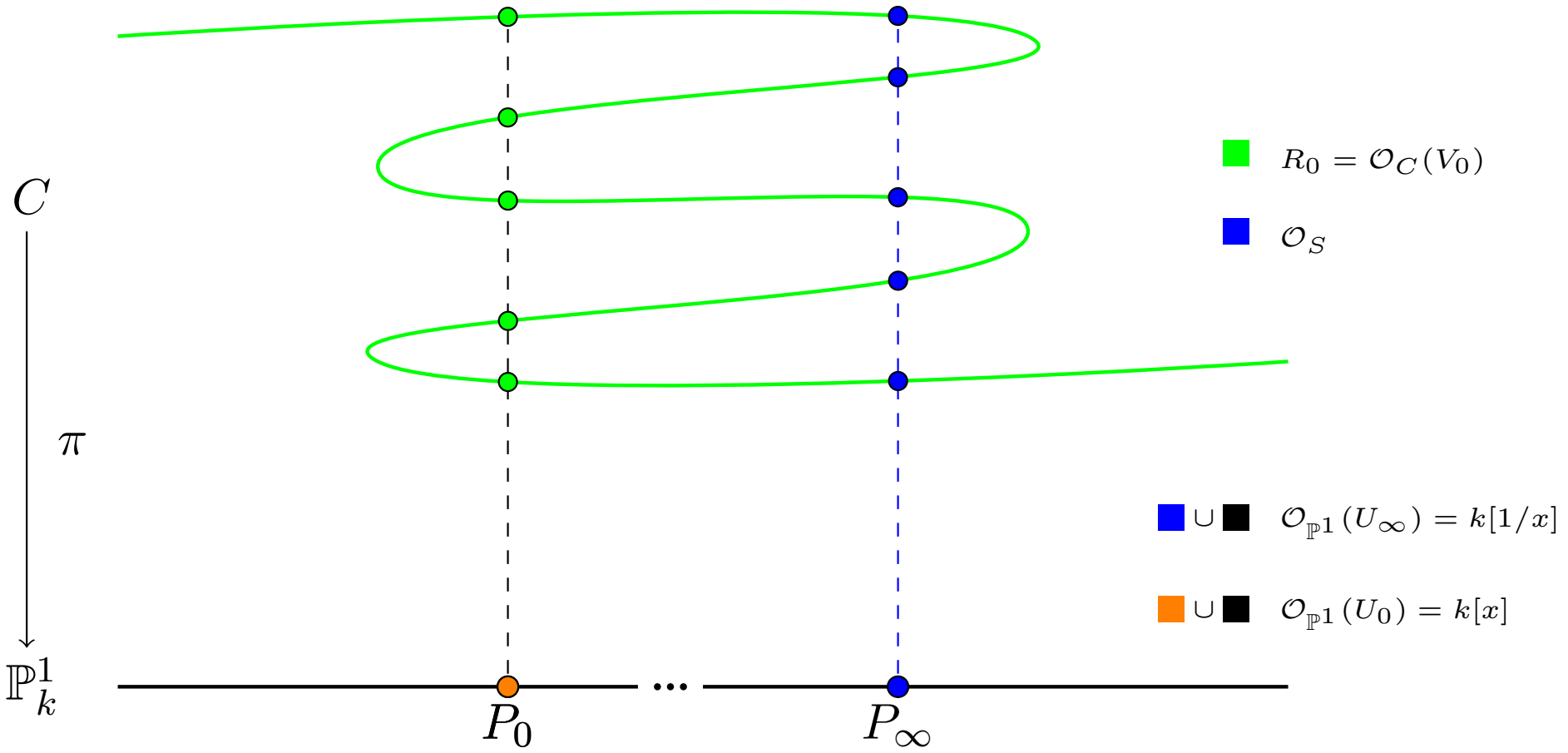

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$\mathcal{F}$ represented by $\mathcal{F}(V_0)$ and $\mathcal{F}(S)$

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Theorem

For all $[\mathcal{F}] \in \mathrm{Pic}^0(C)$ there exists $D \in \mathrm{Div}^0(C)$ s.t.

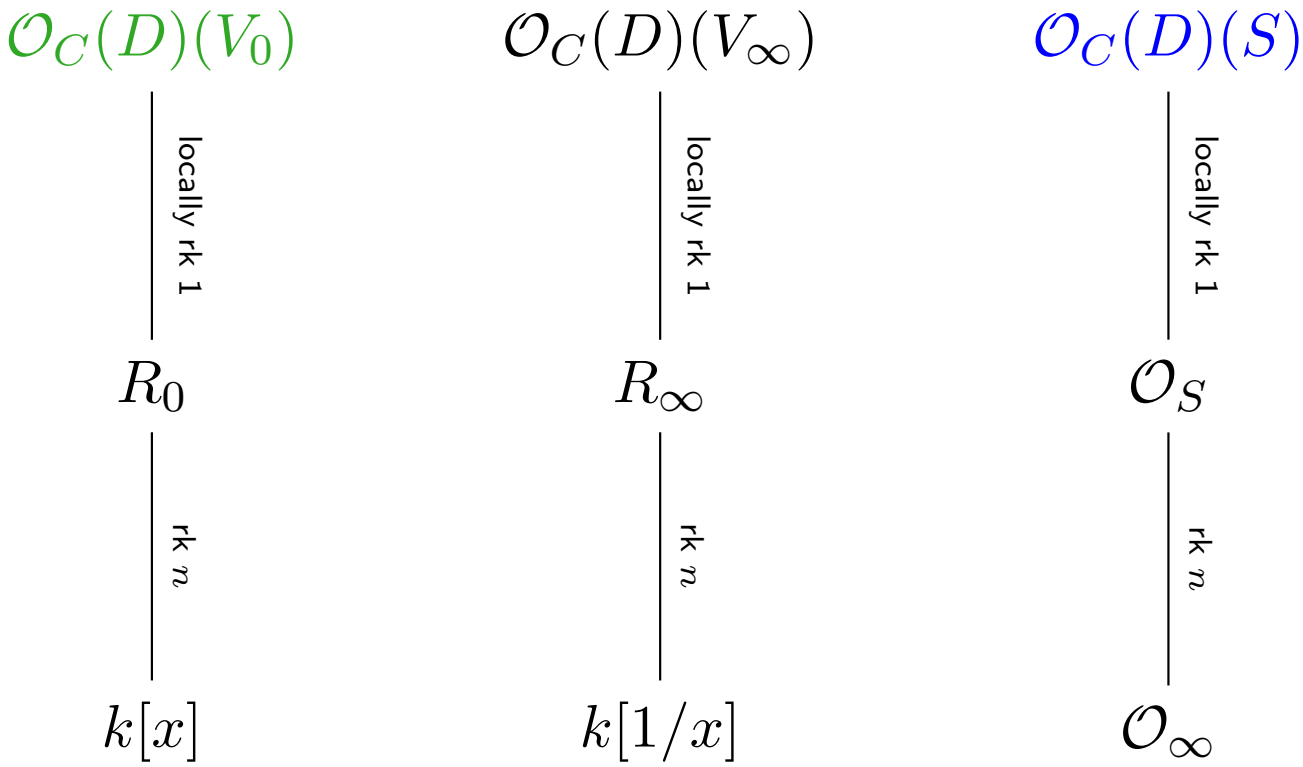
$$[\mathcal{F}] = [\mathcal{O}_C(D)], \quad I = \mathcal{O}_C(D)(V_0) \subseteq R_0, \quad \mathcal{O}_C(D)(S) = x^r\,\mathcal{O}_S$$

$$\deg_k I = rn \leq 2\,g + n$$

This induces an isomorphism

$$\Phi : \mathrm{Pic}^0(C) \to \mathcal{I}_x/\mathcal{P}_x, \quad [\mathcal{O}_C(D)] \mapsto [\mathcal{O}_C(D)(V_0)]$$

From Sheaves to Matrices



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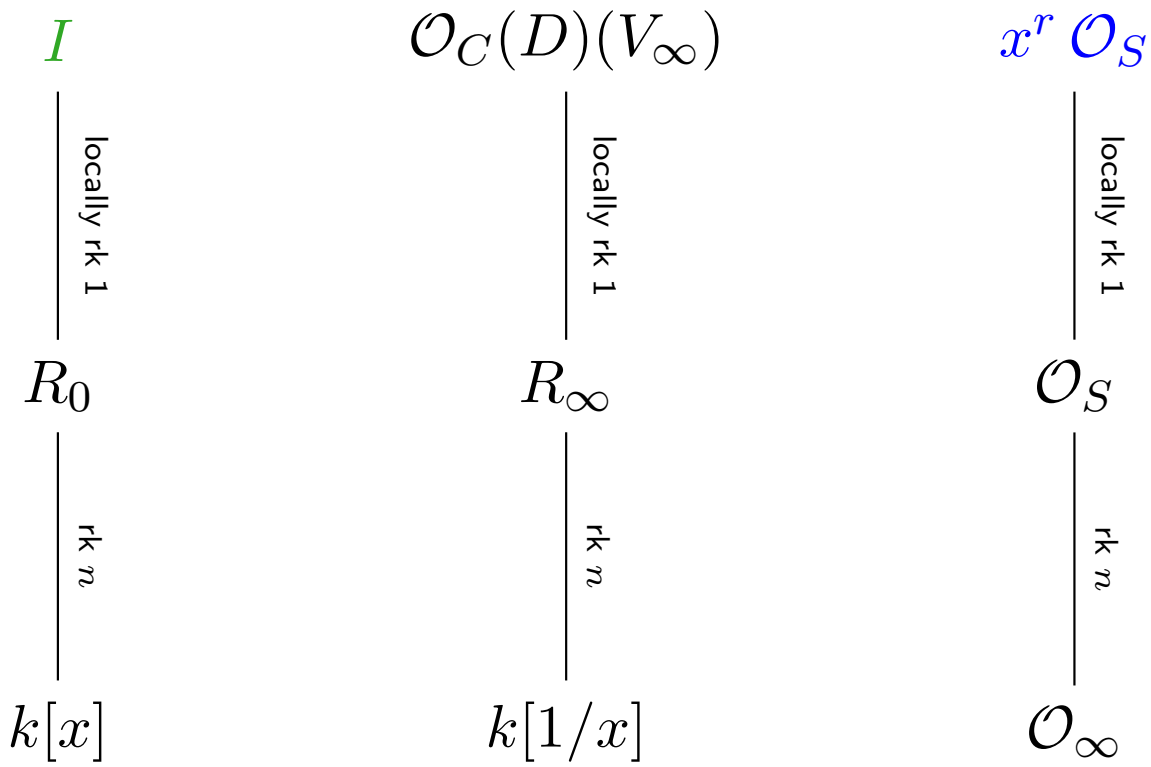

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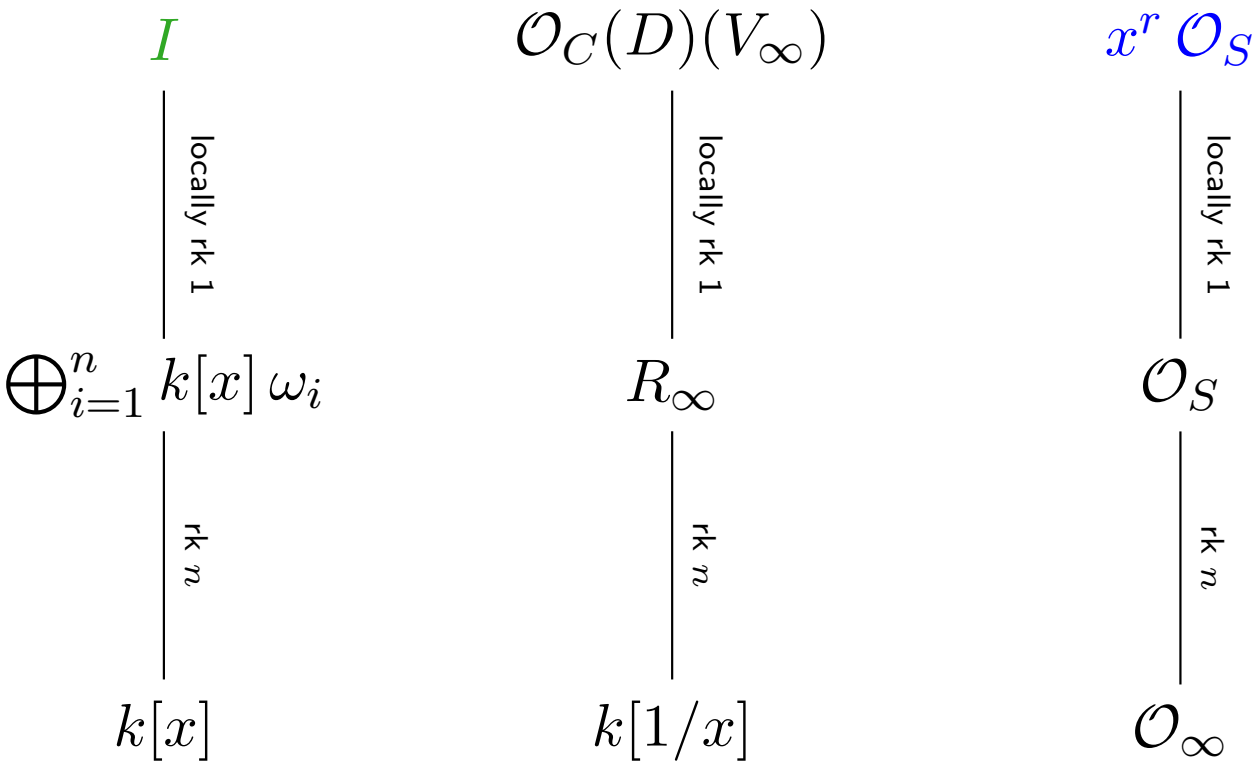

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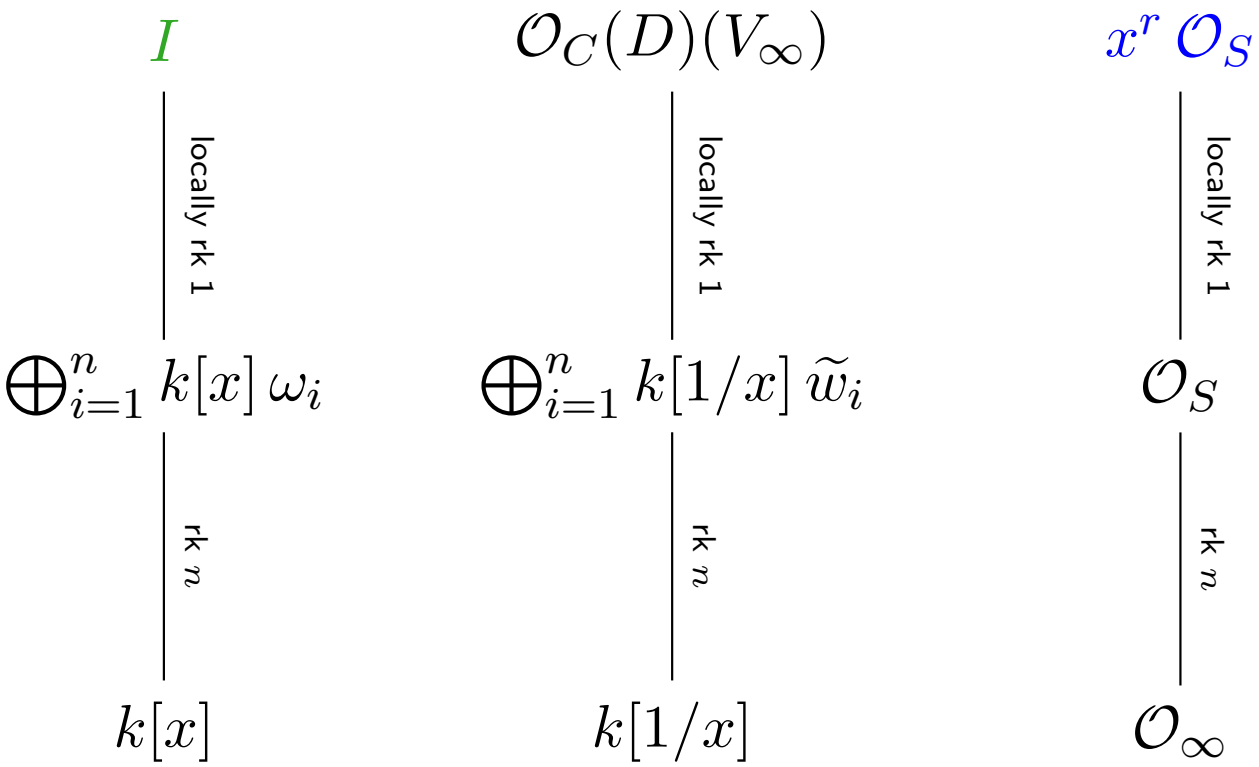

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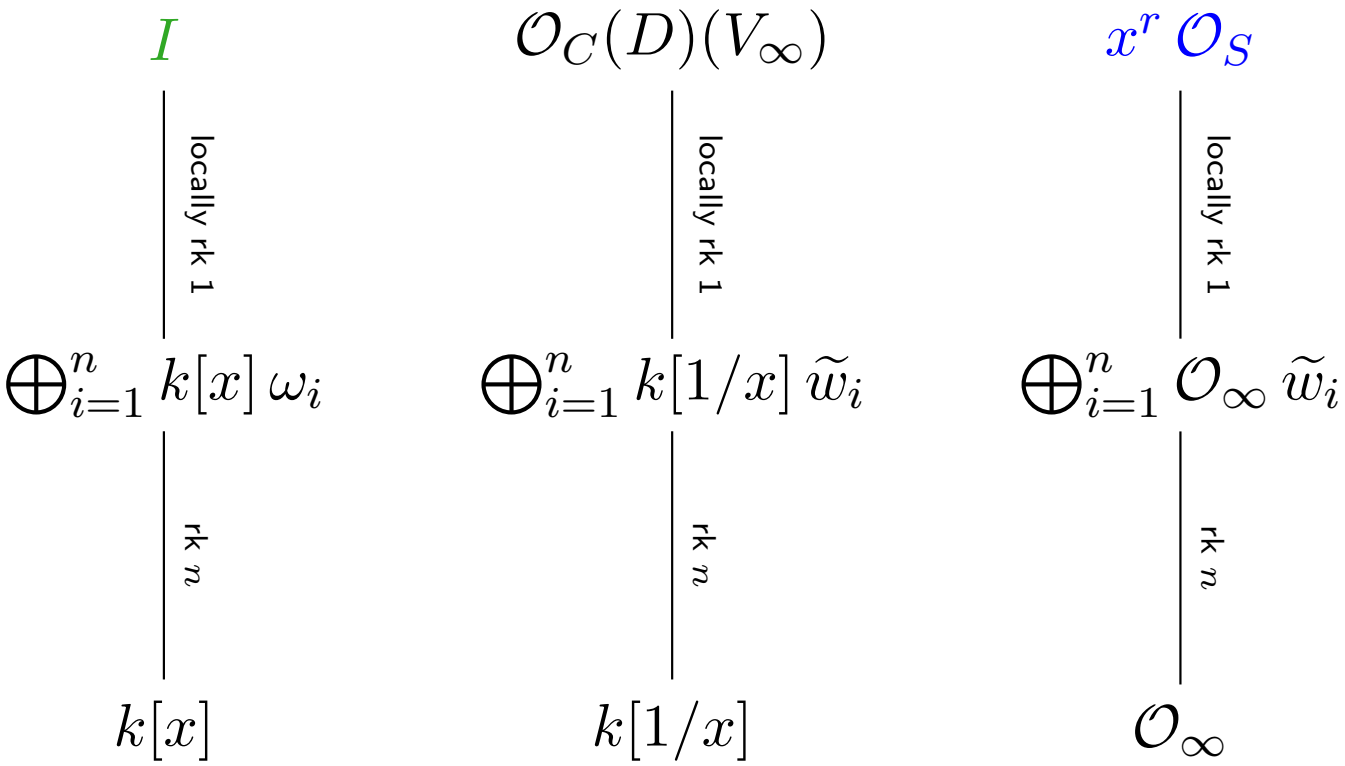

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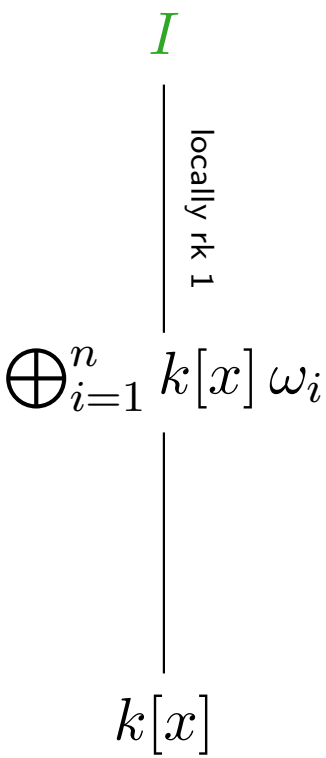

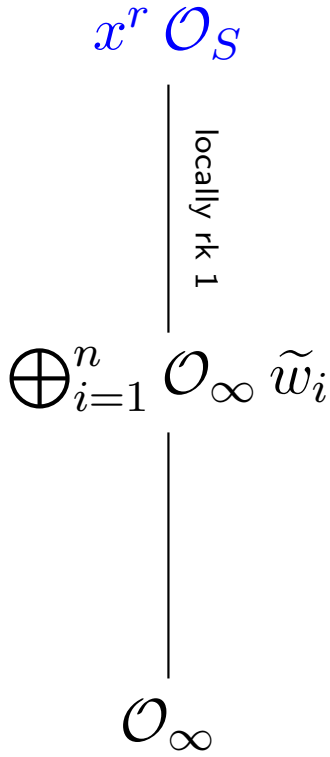

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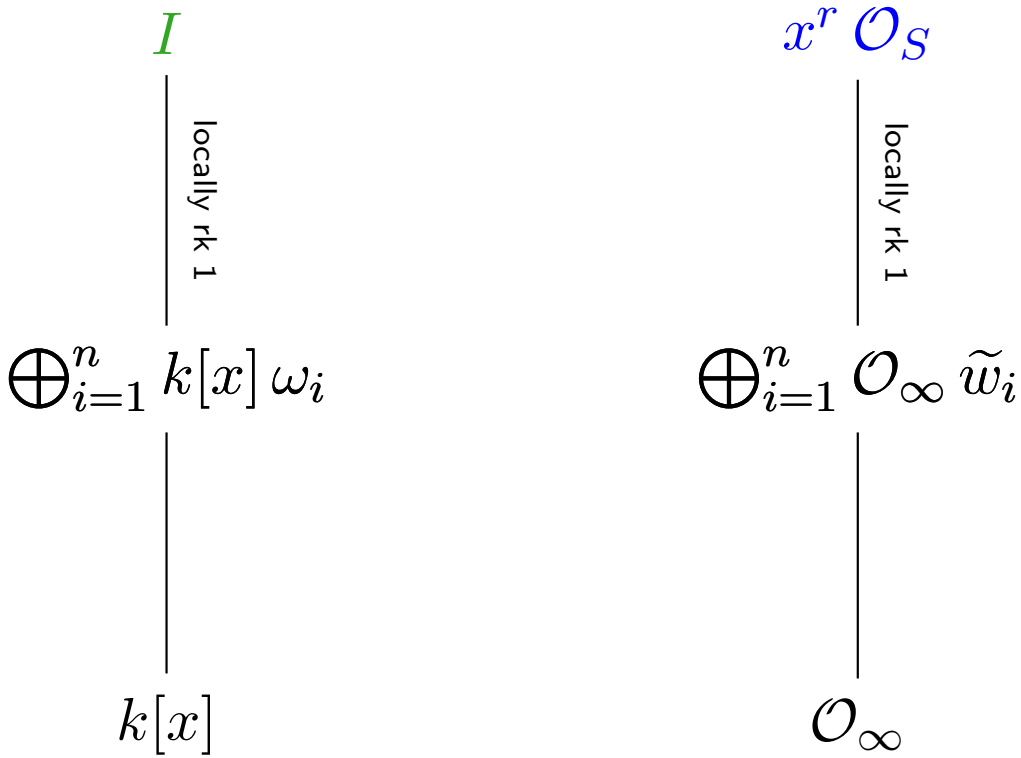

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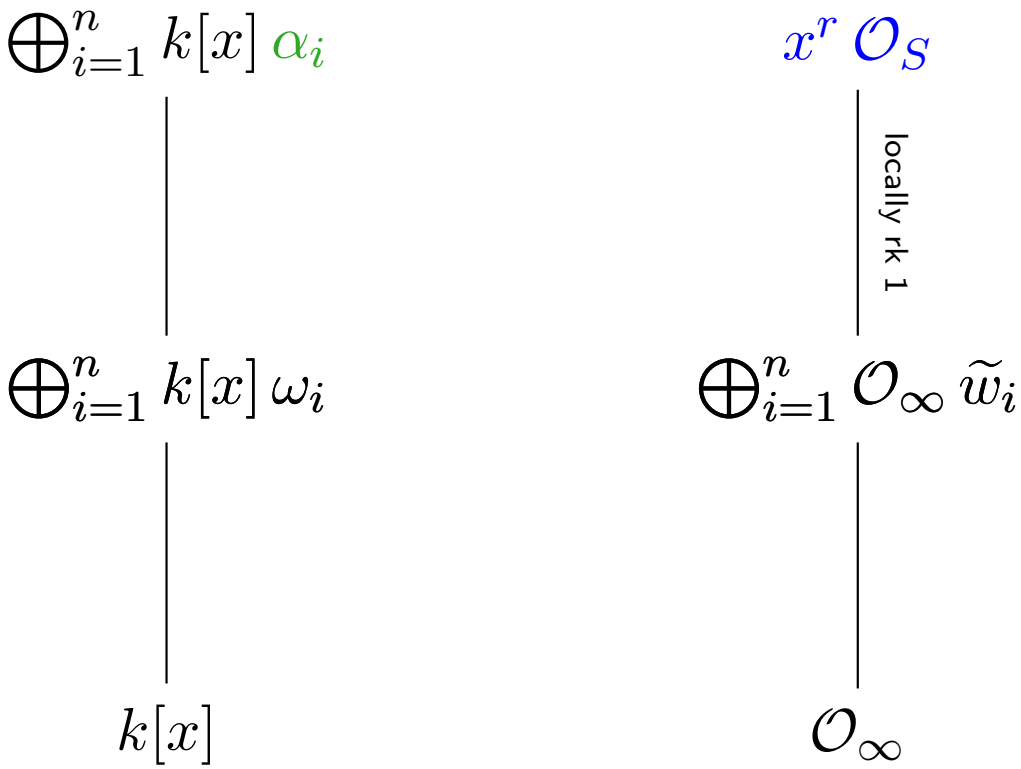

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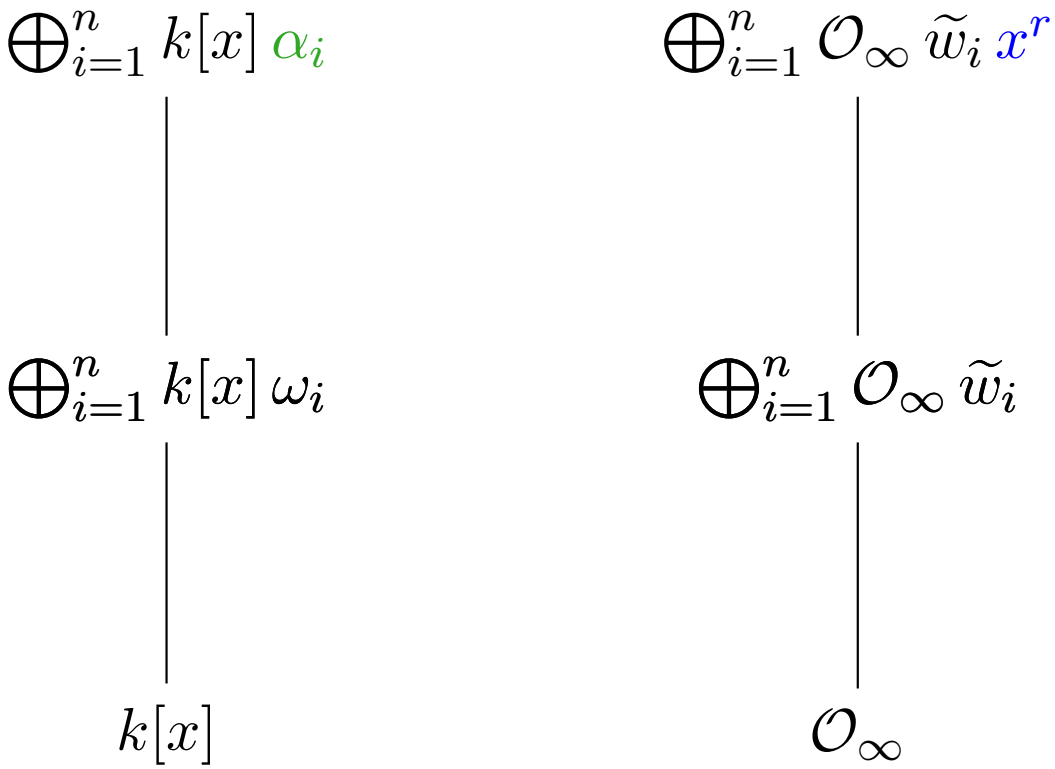

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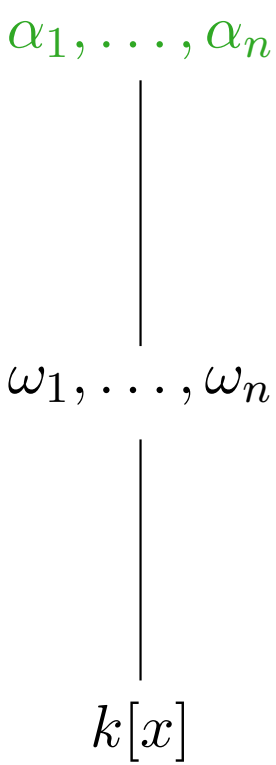

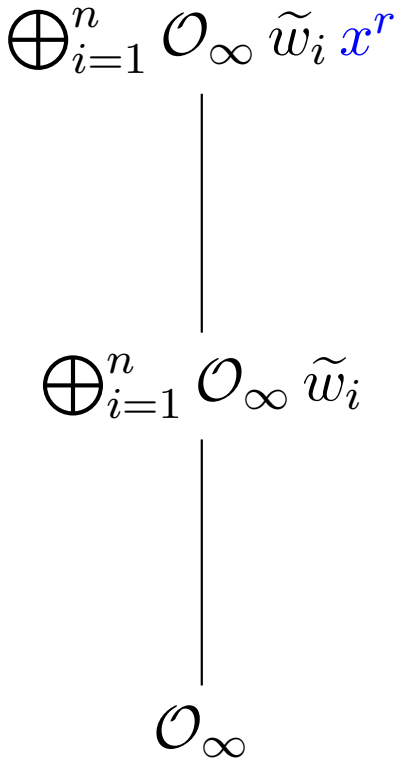

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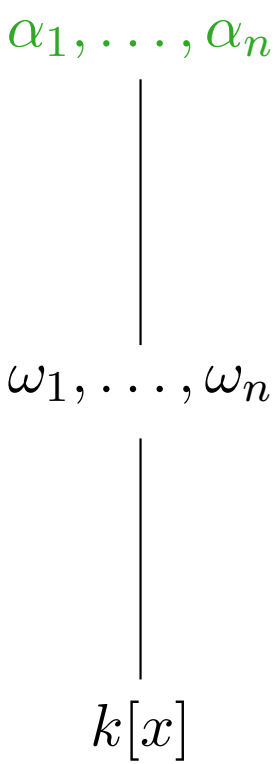

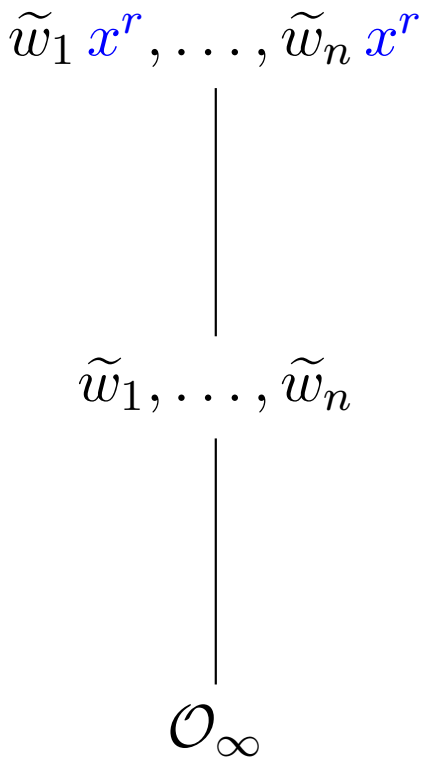

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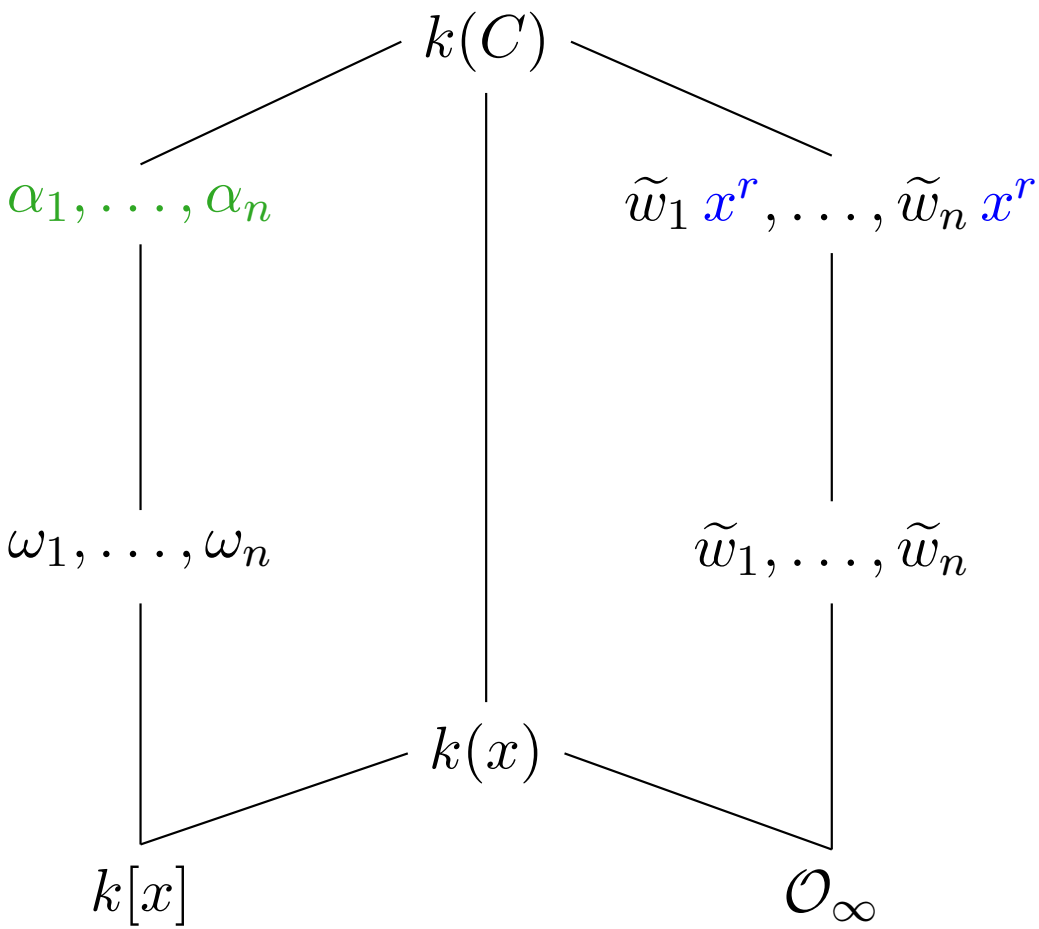

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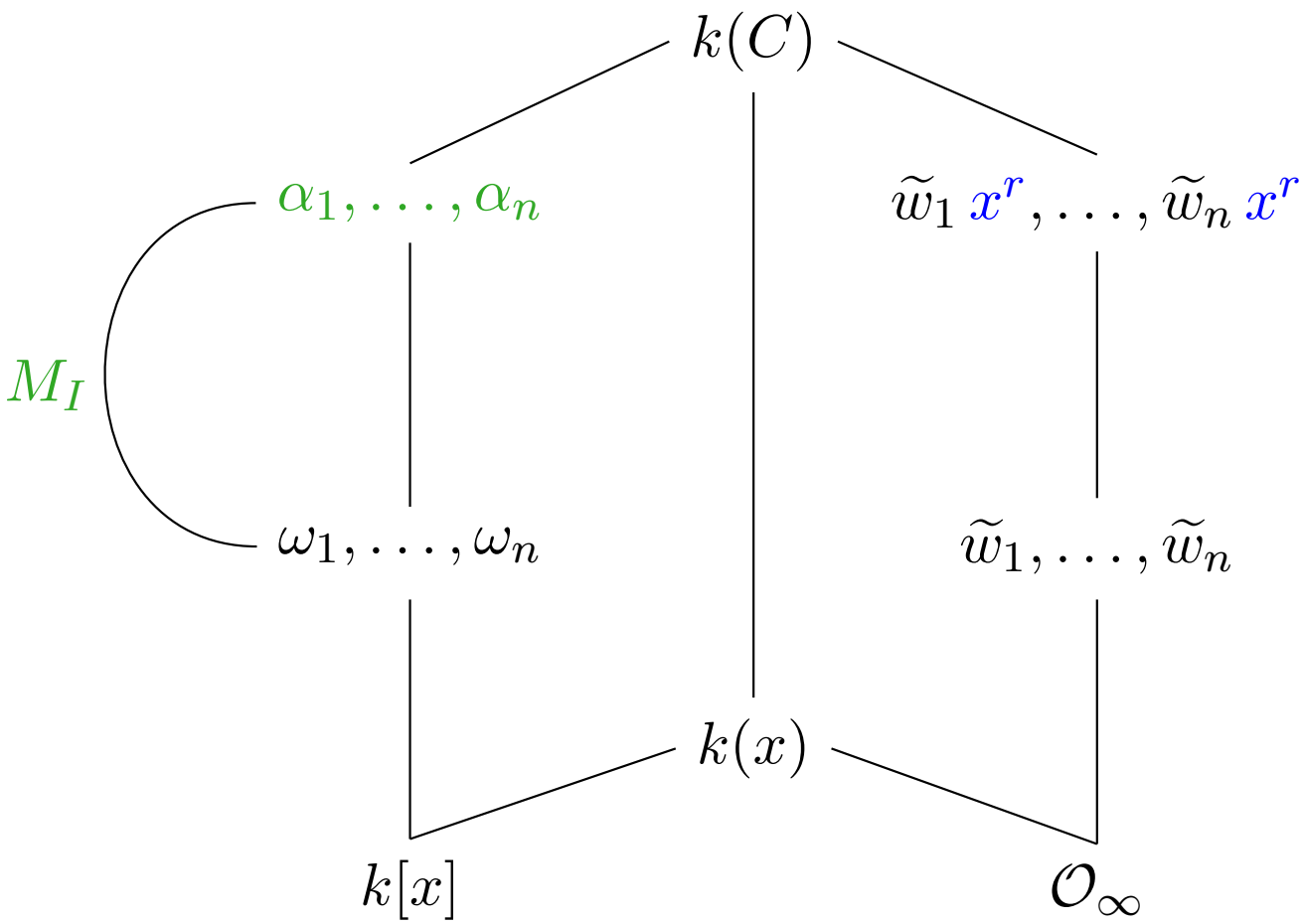

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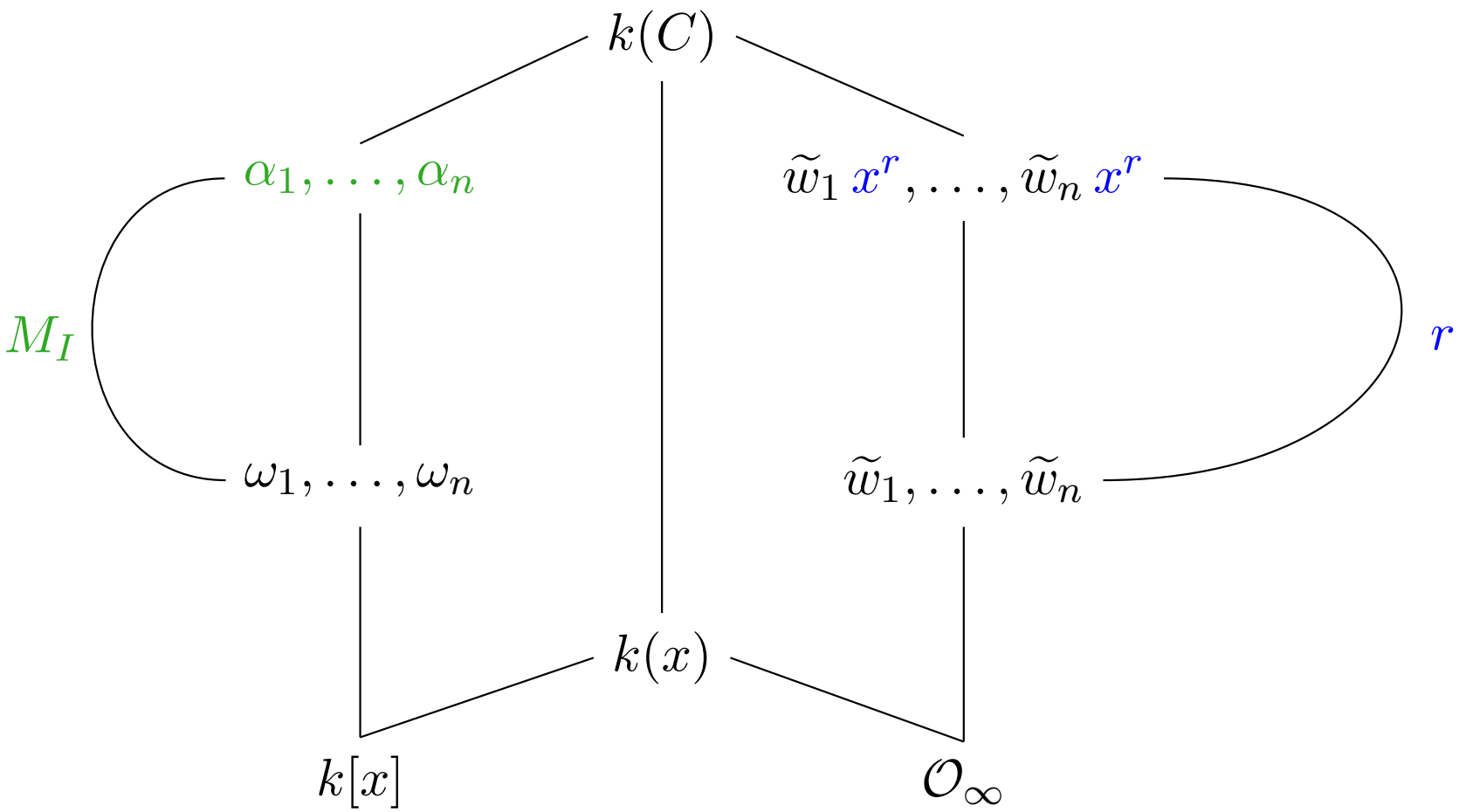

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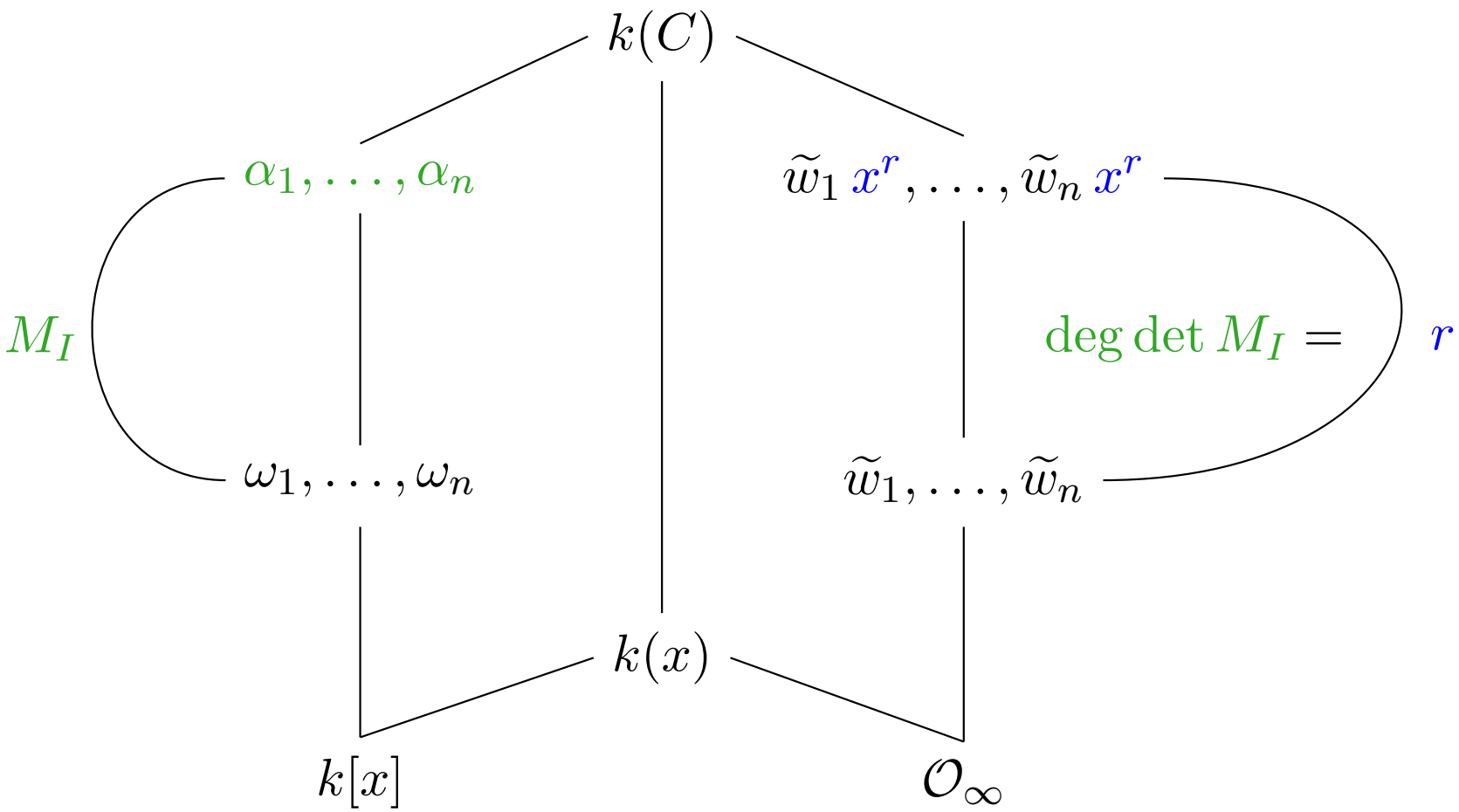

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- Top $(n \times n)$ matrix of $\ker M$ gives basis matrix of J/I

Integral Division

`IntegralDivision`

Input:	$J \subseteq I \subseteq R_0$	given by $M_I,\ M_J \in k[x]^{n\times n}$
Output:	$J/I \subseteq R_0$	given by $M_{J/I} \in k[x]^{n\times n}$

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(1) $(\vec{\beta}_j)_{1\le j\le h} \leftarrow (\sum_{i=1}^{n} \mu_{ij}\ M_I(i))_{1\le j\le h},\ \mu_{ij} \in \{0,1\}$ chosen randomly

(2) $(M_{\beta_1}, \ldots, M_{\beta_h}) \leftarrow (\underline{\texttt{BasMat}}(\vec{\beta}_j))_{1\le j\le h}$

(3) $M_{\ker} \leftarrow \underline{\texttt{MatrixKernel}}(\texttt{BigMatrix}((M_{\beta_1}, \ldots, M_{\beta_h}), M_J))$

(4) $M \leftarrow$ `ReducedBasisMatrix`(Top$(n \times n)$-Matrix of $M_{\ker}$ of Rank n)

(5) If $\deg\det(M) \neq \deg\det(M_J) - \deg\det(M_I)$, then goto (1).

(6) `return` M

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  (2) for computing the standard basis matrix M_f of f s.t.

$$(f\,\omega_1, \ldots, f\,\omega_n) = (\omega_1, \ldots, \omega_n)\,M_f.$$

Ad (1)

ModFct

Input:	I	given by M_I	s.t. $\deg_k I \in O(g)$, $\deg M_I \in O(g/n)$
Output:	f	given by $\vec{f}$	s.t. $f \in I$, $fR_0 \in \mathcal{P}_x$, $\deg M_f \leq \frac{6g+\deg_k I}{n}$
			$\deg_k fR_0 \leq 2g + \deg_k I + n$

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(1) $s \leftarrow \deg \det(M_I)$, $r \leftarrow \lceil (s+2g)/n \rceil$

(2) $M \leftarrow \underline{\texttt{ReduceMatrix}}\left((x^{-|0|_i - s}\delta_{i,j})M_I\right)$,
$(|D_I|_i)_i \leftarrow$ neg. column degrees of M

(3) $N \leftarrow M\,(x^{|D_I|_i}\delta_{i,j})$, $N_\infty \leftarrow (N_{i,j} \bmod P_\infty)_{i,j}$

(4) $(\mu_1, \ldots, \mu_n)^T \leftarrow \texttt{SolveLES}_k\left(UN_\infty, (1, \ldots, 1)^T\right)$

(5) $\texttt{return}\ \vec{f} \leftarrow (x^{-|0|_i}\delta_{i,j})N(\mu_1 x^r, \ldots, \mu_n x^r)^T$

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(1) $T_f \leftarrow \texttt{Reduce}_h(T_h \vec{f})$

(2) $B \leftarrow \texttt{Reduce}_h(T_f \cdot T_h \vec{\omega}_1) \ldots \texttt{Reduce}_h(T_f \cdot T_h \vec{\omega}_n)$

(3) $\texttt{return Reduce}_h(T_h^{-1} \cdot B)$

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(1) $M_f \leftarrow \texttt{ModFctBasMat}(M_I)$

(2) $M \leftarrow \texttt{ReducedBasisMatrix}(M_f \cdot M_J)$

(3) $\texttt{return IntegralDivision}(M, M_J)$

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(1) $|I| \leftarrow$`ReadOffInvariant`(M_I)

(2) If $|I| \geq 0$, then `return` **Yes**

(3) `return` **No**

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- Birkhoff matrix decomposition/Grothendieck:
  $\mathcal{F}$ coherent torsion free sheaf on C:

$$\pi_* \mathcal{F} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}\left(|\mathcal{F}|_i\right).$$

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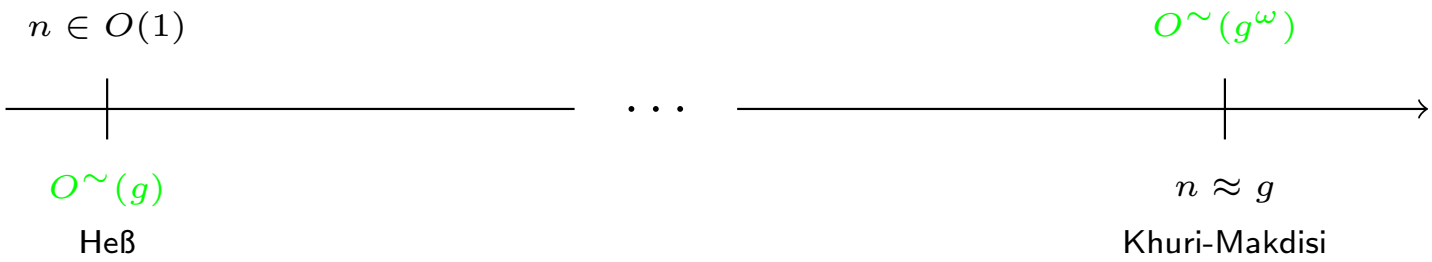

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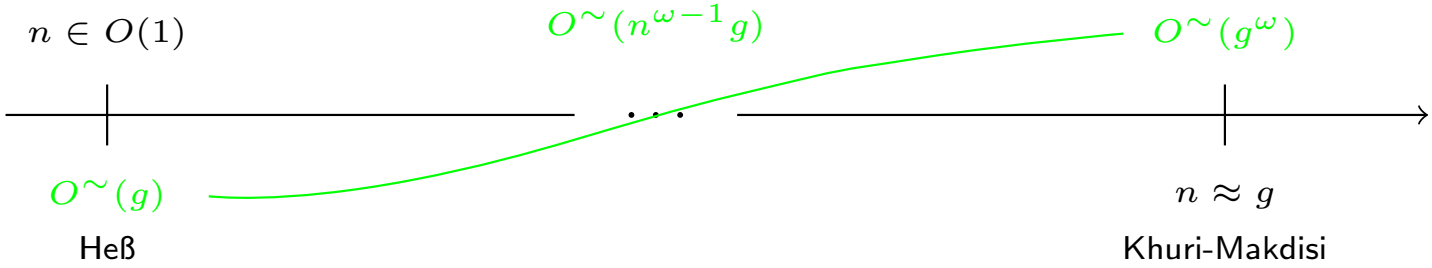

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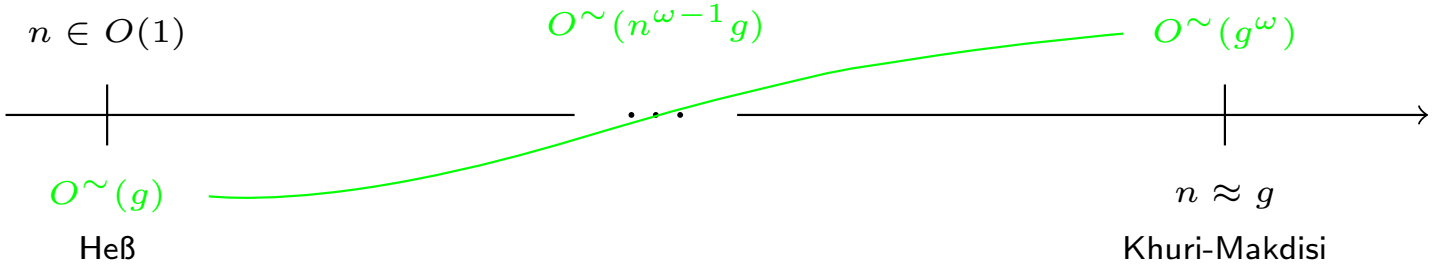

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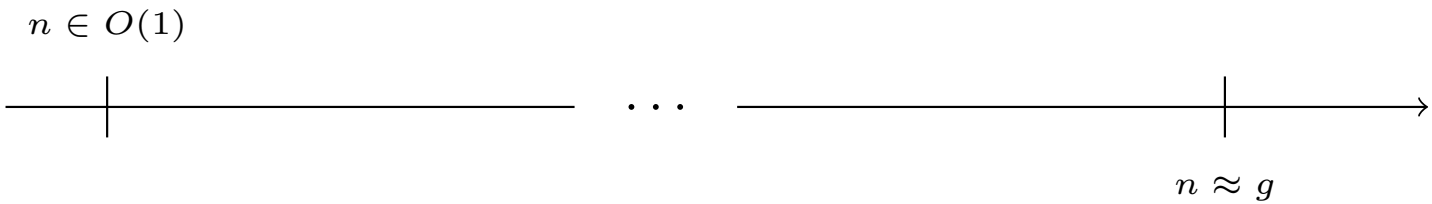

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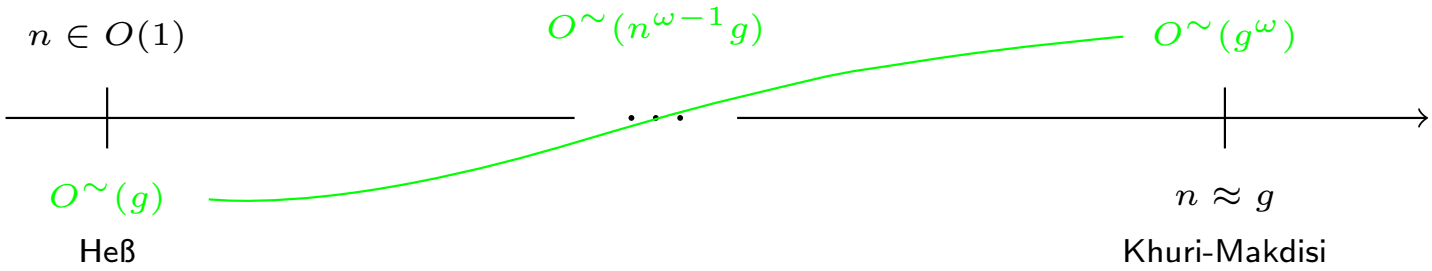

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