Universität Stuttgart

Computations with structures in Lie theory

Computeralgebra-Tagung

Kassel, May 2017

Meinolf Geck

Introduction

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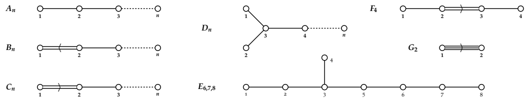
- "Explicit list of groups" known since about 1980.
- 2nd generation proof: D. Gorenstein, R. Lyons, R. Solomon, Math. Survey and Monographs, Amer. Math. Soc., 1994 –?? (currently 6 volumes).

The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ 1 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ [illegible]	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920	C_{11} 11
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A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{\substack{i=1\\ i\neq 2,8}}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O^+_{2n}(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O^-_{2n}(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

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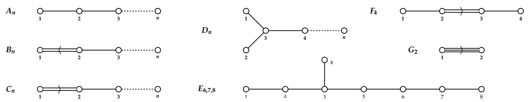

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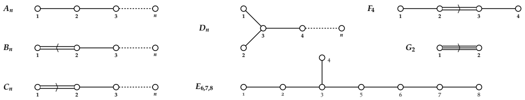
Sz

44

The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ **1** 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											${}^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	${}^2D_4(2^2)$ 197 406 720	$G_2(2)'$ ${}^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	${}^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	${}^2D_4(3^2)$ 10 151 968 619 520	${}^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	${}^3D_4(2^3)$ 211 341 312	${}^2E_6(2^2)$ 76 532 479 683 774 853 939 200	${}^2B_2(2^3)$ 29 120	Tits* ${}^2F_4(2)'$ 17 971 200	${}^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	${}^2D_5(2^2)$ 25 015 379 558 400	${}^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ 18 830 052 912 953 932 311 099 032 439 972 660 332 140 886 784 940 152 038 522 449 391 826 616 580 150 109 878 711 243 949 982 163 690 448 626 420 940 800 000	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	${}^3D_4(3^3)$ 20 560 831 566 912	${}^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	${}^2B_2(2^5)$ 32 537 600	${}^2F_4(2^3)$ 264 905 352 699 586 176 614 400	${}^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	${}^2D_4(4^2)$ 67 536 471 195 648 000	${}^2A_3(9)$ 3 265 920	C_{11} 11
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ 111 131 458 114 940 385 379 597 233 477 884 941 280 664 199 527 155 056 307 251 745 263 504 588 800 000 000	$E_8(4)$ 191 797 292 142 671 717 784 639 787 897 912 906 421 357 507 604 216 557 533 558 287 598 236 977 154 127 870 984 484 770 345 340 348 298 409 697 395 609 822 849 492 217 656 441 474 908 160 000 000 000	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	${}^3D_4(4^3)$ 67 802 350 642 790 400	${}^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	${}^2B_2(2^7)$ 34 093 383 680	${}^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	${}^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	${}^2D_4(5^2)$ 17 880 203 250 000 000 000	${}^2A_2(64)$ 5 515 776	C_{13} 13
A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{i=1, i\neq 2,8}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	${}^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	${}^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	${}^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	${}^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	${}^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^+(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^-(q)$ ${}^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ ${}^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

- Alternating Groups
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- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates[†] Symbol Order[‡]

M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1), J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS, O\text{–}S$ $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi'_{24} 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

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HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM 4 030

F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi'_{24} 1 255 205 709 190 661 721 292 800	F_2 4 154 78 191 177 58

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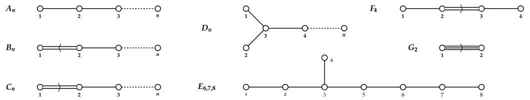

$0, C_1, \mathbb{Z}_1$ **1** 1																	**C_2** 2
$A_1(4), A_1(5)$ **A_5** 60	$A_2(2)$ **$A_1(7)$** 168											$^2A_3(4)$ **$B_2(3)$** 25 920	**$C_3(3)$** 4 585 351 680	**$D_4(2)$** 174 182 400	**$^2D_4(2^2)$** 197 406 720	$G_2(2)'$ **$^2A_2(9)$** 6 048	**C_3** 3
$A_1(9), B_2(2)'$ **A_6** 360	$^2G_2(3)'$ **$A_1(8)$** 504											**$B_2(4)$** 979 200	**$C_3(5)$** 228 501 000 000 000	**$D_4(3)$** 4 952 179 814 400	**$^2D_4(3^2)$** 10 151 968 619 520	**$^2A_2(16)$** 62 400	**C_5** 5
A_7 2 520	**$A_1(11)$** 660	**$E_6(2)$** 214 841 575 522 005 575 270 400	**$E_7(2)$** 7 997 476 042 075 799 759 100 487 262 680 802 918 400	**$E_8(2)$** 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	**$F_4(2)$** 3 311 126 603 366 400	**$G_2(3)$** 4 245 696	**$^3D_4(2^3)$** 211 341 312	**$^2E_6(2^2)$** 76 532 479 683 774 853 939 200	**$^2B_2(2^3)$** 29 120	Tits* **$^2F_4(2)'$** 17 971 200	**$^2G_2(3^3)$** 10 073 444 472	**$B_3(2)$** 1 451 520	**$C_4(3)$** 65 784 756 654 489 600	**$D_5(2)$** 23 499 295 948 800	**$^2D_5(2^2)$** 25 015 379 558 400	**$^2A_2(25)$** 126 000	**C_7** 7
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$+1)$ $-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^+(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^-(q)$ ^2D_n $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}$

Character table $\mathfrak{X}(G)$ of G (Frobenius ~1896)

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- Size $r \times r$ where $r =$ number of conjugacy classes of G.
- Entries are algebraic integers.

$\mathfrak{A}_5$	()	(12)(34)	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
χ_3	3	-1	0	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

- Contains subtle information about G in a compact form.

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Application: Existence of certain subgroups.

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χ_2	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
χ_3	3	-1	0	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

- Contains subtle information about G in a compact form.

Example: class structure constants

Let C_1, C_2, C_3 be conjugacy classes of G. Then the number of pairs $(x, y) \in C_1 \times C_2$ with $xy \in C_3$ can be computed from $\mathfrak{X}(G)$.

Application: Existence of certain subgroups. There is $U \leqslant G$ with $U \cong \mathfrak{A}_5$, if there are $x, y \in G$ with $o(x) = 2$, $o(y) = 3$, $o(xy) = 5$. (Then set $U = \langle x, y \rangle$.)

Character table $\mathfrak{X}(G)$ of G (Frobenius ∼1896)

- Size $r \times r$ where $r =$ number of conjugacy classes of G.
- Entries are algebraic integers.

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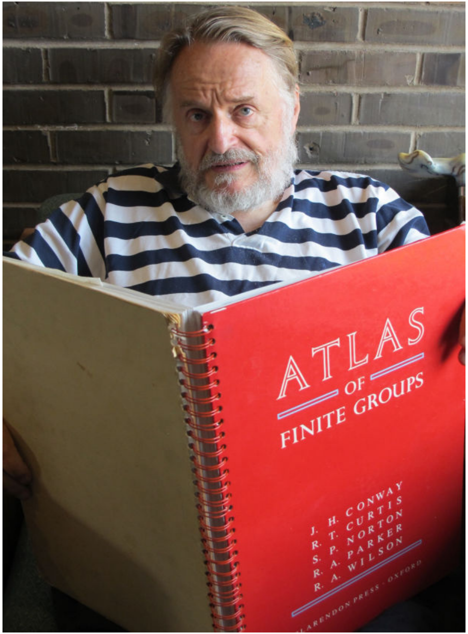
Lusztig (2002): The simple group $E_8(q)$ contains subgroups $\cong \mathfrak{A}_5$.



The ATLAS contains $\mathfrak{X}(G)$ for

- all sporadic simple G

  incl. $G =$ Monster with $|M| \approx 10^{54}$,

  $\mathfrak{X}(G)$ of size 194×194
- $G = \mathfrak{A}_n$ where $5 \leqslant n \leqslant 13$
- Chevalley groups $G = G(q)$ with small q

  e.g., ${}^2G_2(27)$, $O_8^{\pm}(2)$, $O_8^{\pm}(3)$, $F_4(2)$, ${}^2E_6(2)$

Also information about:

- various constructions of the groups
- maximal subgroups
- central extensions and automorphisms

...

All this electronically available in GAP !

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Use methods from Lie theory and algebraic geometry !

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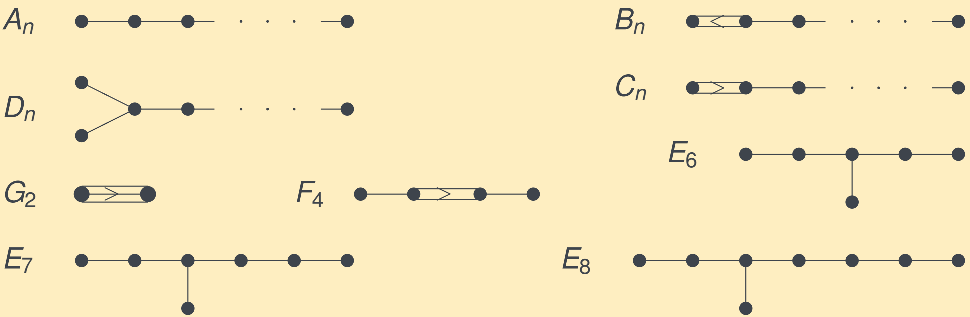

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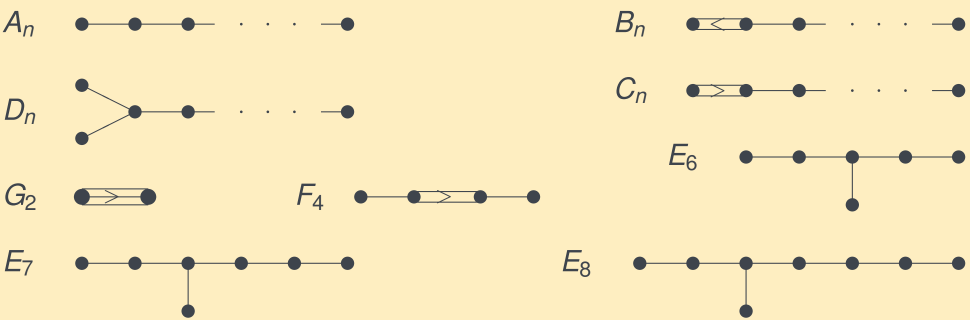

$\rightsquigarrow$ the 16 infinite families of groups of Lie type in the “periodic table”:

$$A_{n-1} \leftrightarrow \mathrm{SL}_n(q), \quad {}^2A_{n-1} \leftrightarrow \mathrm{SU}_n(q), \quad B_n \leftrightarrow \mathrm{SO}_{2n+1}(q), \ \ldots, \quad \mathrm{E}_8(q)$$

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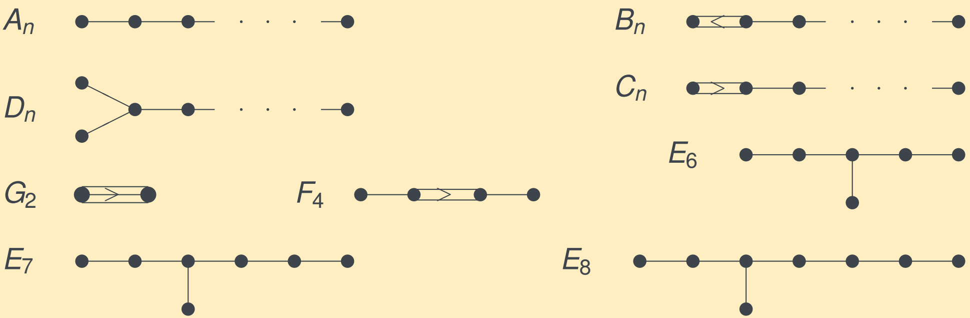

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Aim: uniform — “generic” — description of the tables $\mathfrak{X}(\mathbf{G}^F)$.

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Class types: $A_1 : \begin{pmatrix} \rho^a & 0 \\ 0 & \rho^a \end{pmatrix}$, $A_2 : \begin{pmatrix} \rho^a & 0 \\ 1 & \rho^a \end{pmatrix}$, $A_3 : \begin{pmatrix} \rho^a & 0 \\ 0 & \rho^b \end{pmatrix}_{a \neq b}$, $B_1 : \begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^{aq} \end{pmatrix}_{a \neq \text{mult.}(q+1)}$.

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element	number of classes	number of elements in each class
A_1	$q-1$	1
A_2	$q-1$	$(q-1)(q+1)$
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“Generic charater table” of $\mathrm{GL}_2(q)$ (R. Steinberg, 1951)

		A_1	A_2	A_3	B_1	
$\chi_1^{(n)}$	$n{=}1,2,\ldots,q-1$	ε^{2na}	ε^{2na}	$\varepsilon^{n(a+b)}$	ε^{na}	$(\varepsilon^{q-1}=1)$
$\chi_q^{(n)}$	$n{=}1,2,\ldots,q{-}1$	$q\varepsilon^{2na}$	0	$\varepsilon^{n(a+b)}$	$-\varepsilon^{na}$	$(\varepsilon^{q-1}=1)$
$\chi_{q+1}^{(m,n)}$	$m,n{=}1,2,\ldots,q-1$ $m\neq n$; $(m,n)\equiv(n,m)$	$(q{+}1)\varepsilon^{(m+n)a}$	$\varepsilon^{(m+n)a}$	$\varepsilon^{ma+nb}+\varepsilon^{na+mb}$	0	$(\varepsilon^{q-1}=1)$
$\chi_{q-1}^{(n)}$	$n{=}1,2,\ldots,q^2-2$ $n\neq \mathrm{mult.}(q+1)$	$(q{-}1)\delta^{na(q+1)}$	$-\delta^{na(q+1)}$	0	$-(\delta^{na}+\delta^{naq})$	$(\delta^{q^2-1}=1)$

Lusztig’s geometric character theory ($\sim$1975–today)

Deligne–Lusztig (1976):
  Construct characters
  using ℓ-adic cohomology

Lusztig (1984–today):
  Comprehensive theory
  “Perverse sheaves”
  $\updownarrow$
Multiple applications:
  character formulae,
  canonical bases, . . .
(see ICM talk 1990)

Let $\mathrm{CF}(\mathbf{G}^F)$ be the vector space of all functions $f\colon \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ ($\ell \neq p$), which are constant on the conjugacy classes of $\mathbf{G}^F$.

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Aim: Bring this theory on the computer !
And use this to solve open conjectures on characters of finite groups.

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Let V be a $\mathbb{Q}$-vector space with basis $\{\alpha_t \mid t \in S\}$. For $s \in S$ define

$$w_s \colon V \to V, \qquad \alpha_t \mapsto \alpha_t - c_{st}\alpha_s \qquad (w_s^2 = \mathrm{id}_V).$$

- Weyl group $\mathbf{W} = \langle w_s \mid s \in S\rangle \subseteq \mathrm{GL}(V)$.
- Root system $\Phi = \{w(\alpha_s) \mid w \in \mathbf{W}, s \in S\}$.

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Definition.

A root datum is a quadruple $(X, \Phi, Y, \Phi^\vee)$ such that:

- ...
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To give a root datum of type $C = (c_{st})_{s,t\in S}$ is the same as to specify a factorisation $C = \breve{A} \cdot A^{\mathrm{tr}}$ where A and $\breve{A}$ are integer matrices with rows indexed by S and columns indexed by another index set I with $|I| \geqslant |S|$.

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CHEVIE project: M.Geck, G.Hiss, F.Luebeck, G.Malle, J.Michel, G.Pfeiffer

http://www.math.rwth-aachen.de/~CHEVIE/

Implemented in GAP3; now also made available for 64-bit machines by J.Michel:

http://webusers.imj-prg.fr/~jmichel/gap3

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  - Very twisted example:

    $P_0 = \begin{pmatrix} 0 & \sqrt{2}^{-1} \\ \sqrt{2} & 0 \end{pmatrix}$ $\rightsquigarrow$ $\{\mathrm{Suz}(q) \mid q = \sqrt{2}^{2m+1}, m \geqslant 0\}$.

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- The conjugacy classes of semisimple elements of $\mathbf{G}(q)$

  (parametrisation, root data of centralisers, etc.)
- The head of the "generic" character table of $\mathbf{G}(q)$.

  (Semisimple "class types" + data base for unipotent classes.)
- The list of character degrees $\chi(1)$, $\chi \in \mathrm{Irr}(\mathbf{G}(q))$.

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E.g., for $E_8(q)$, $q \equiv 1 \bmod 60$, the generic character table has size 10061×9868;

largest entry in this table: sum of 696,729,600 roots of unity.

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